MATH 140 CALCULUS I
REVIEW PROBLEMS for FINAL EXAM

In these four pages you will find a set of problems taken from previous Final Exams in Math 140. The coming Final Exam will mainly include problems which are more or less similar in type and content to any of the problems in these four pages; a few problems of other sorts may possibly appear as well. The problems on the last two pages have been arranged as on a typical Math 140 Final Exam, so that you may know the format, length, and level of difficulty to expect for this semester’s Final Exam.
The Final Exam may include some problems which are suitable for Math 140 but are not exactly like any of the problems here.

1. a) Evaluate the limits, using $+\infty$ or $-\infty$ where appropriate. Explain your answers.

\[
\begin{align*}
\text{i)} & \quad \lim_{x \to -3} \frac{x^2 - 2x - 3}{x^2 - 3x} \\
\text{ii)} & \quad \lim_{x \to -3} \frac{x^2 - 2x - 3}{x^2 - 3x} \\
\text{iii)} & \quad \lim_{x \to -\infty} \frac{5x^2 + 1}{2x + 3} \\
\text{iv)} & \quad \lim_{x \to 2} \frac{|x - 2|}{x - 2} \\
\text{v)} & \quad \lim_{x \to \frac{\pi}{2}} \frac{1}{x - \frac{\pi}{2}} - \tan x \\
\text{vi)} & \quad \lim_{h \to 0} \frac{\cos \theta - \sin h}{h} \\
\end{align*}
\]

Ans. i) 4/3; ii) 2/3; iii) $-\infty$; iv) 1; v) +\infty; vi) $\cos \theta$

b) Draw a graph illustrating your answers to i) and ii) above.

Ans. $y = \frac{x + 1}{x}$ with a hole at $(3, \frac{4}{3})$

2. Determine the absolute maximum and minimum values of $f(x) = 2 \cos x - \cos 2x$, for $0 \leq x \leq \pi$.

Ans. There is one (interior) critical point, $\pi/3$, and the ends are 0 and $\pi$. Now $f(0) = 1, f(\pi/3) = 3/2, f(\pi) = -3$; whence $3/2$ is the maximum and $-3$ is the minimum.

3. Sketch the graph of $y = \frac{3x^2 - 1}{x^3} = \frac{3}{x} - \frac{1}{x^3}$. Find and show domain, symmetry, asymptotes, intercepts, extremes, and inflection points.

Ans. 

- dom $x \neq 0$; origin sym; asys = axes; ints $\left(\pm \frac{1}{\sqrt{3}}, 0\right)$; loc max (1, 2), loc min (-1, -2); inflection points $(\pm \sqrt{2}, \pm 5\sqrt{2}/4)$

4. A factory makes cylindrical cans, with cardboard sides costing 2¢/in² and metal ends costing 8¢/in². Assume that all cans contain the same volume $V$. Determine the ratio of height $h$ to radius $r$ of the cans so that the cost $C$ of materials for a can may be a minimum. Explain why the cost is least for the proportions you find.

Ans. When $r^3 = \frac{V}{5\pi}$, $C'(r) = 0$ and $C'' > 0$, so min; then $h/r = 8$.

5. A rectangular garden is to be laid out along a neighbor's lot and is to contain 432 square yards. If the neighbor pays for half the dividing fence, what should the dimensions of the garden be so that the cost to the owner of fencing it in may be a minimum?

Ans. 18 yards by 24 yards

6. Consider the function $f(x) = 6x^5 + 20x^3 + 75x + 1$.

a) Show that $f'(x) > 0$ for all $x$. Ans. $f'$ is a sum of nonnegative and positive terms.

b) Conclude that $f(x_0) = 0$ for exactly one real number $x_0$. Ans. By a), $f$ is increasing; therefore no value is repeated. $f$ is continuous (poly) and $f(-1) < 0 < f(0)$, so $f$ has a zero, by IVT.

c) Show that $x_0$ lies to the left of 0.

Ans. $f$ is increasing and $f(0) > 0$.

7. A particle moves along a line so that at time $t$ its acceleration is $a(t) = \sin t$. At time $t = 0$, its position is $s = 2$ and it is not moving.

a) Find the position of the particle as a function of time.

Ans. $s(t) = -\sin t + t + 2$

b) Assuming $t > 0$, where is the particle the first time it stops moving?

Ans. $s = 2\pi + 2$

8. A car going along at 30 mi/hr (=44 ft/sec) begins to decelerate at the constant rate of 12 ft/sec². How far does the car travel while decelerating, before it stops?

Ans. 80 2/3 feet

9. Differentiate:

a) $f(x) = \int_0^x \tan(t^3) \, dt$ 

b) $g(x) = \int_0^x \tan(t^3) \, dt$

c) $h(x) = \int_0^x x^2 \tan(t^3) \, dt$

Ans. a) $f'(x) = \tan(x^3)$; b) $g'(x) = 3x^2 \tan(x^3)$; c) $h'(x) = 2x f(x) + x^2 \tan(x^3)$.
10. Air is pumped into a balloon at the rate of \(5t^2\) cubic inches each second, where \(t\) denotes time. What volume of air enters the balloon from the time \(t = 1\) to \(t = 4\)?

Ans. Set \(V(t) = \text{vol at time } t\), then \(V(4) - V(1) = \int_1^4 V'(t) \, dt = \int_1^4 5t^2 \, dt = 15/4 \text{ in}^3\)

11. Two sides of a triangle have lengths 4 feet and 7 feet. The length between them is increasing at the rate of 0.03 rad/sec. How fast is the length of the third side increasing when the angle between the first two sides is \(\pi/3\)?

Law of Cosines. \(c^2 = a^2 + b^2 - 2ab \cos C\)

Ans. By Law of Cos, get \(c = 1.22 \text{ ft/sec}\).

12. Define function \(f\) by \(f(x) = 0\) for \(x < 0\) and for \(2 < x\); and \(f(x) = x\) for \(0 \leq x \leq 1\), \(f(x) = 2 - x\) for \(1 < x \leq 2\). Define function \(g\) by \(g(x) = \int_0^x f(t) \, dt\).

a) Find an expression for \(g\) similar to the one for \(f\).

Ans. \(g(x) = 0\) for \(x < 0\), \(\frac{1}{2} x^2\) for \(0 \leq x \leq 1\), \(2x - \frac{1}{2} x^2 - 1\) for \(1 < x \leq 2\), \(= 1\) for \(2 < x\).

b) Sketch the graphs of \(f\) and \(g\).

c) Where is \(f\) differentiable; where is \(g\) differentiable?

Ans. By the FTC1, \(g\) is differentiable everywhere; \(f\) is differentiable except at 0, 1, 2.

13. A lighthouse has a revolving light which turns at the rate of 3 revolutions per minute. The lighthouse is situated 1/2 mile from a straight beach. Find how fast the spot of light from the beam is moving along the beach when the spot is 1 mile from the point of the beach nearest the light.

Ans. \(\tan \theta = 2x\) and \(\theta = 6\pi\), so \(x = 15\pi\) miles per minute (fast!)

14. A function \(E(x)\) is defined by the integral

\[E(x) = \int_0^x e^{-t^2} \, dt.\]

a) Evaluate; explain each answer.

i) \(E'(x), \quad ii) E''(x), \quad iii) E(0), \quad iv) E'(0)\),

Ans. i) \(E'(x) = e^{-x^2}\) by FTC1; ii) \(E''(x) = -2xe^{-x^2}\)

by power rule \((u^v)' = u^v(\ln u)'\), etc; iii) \(E(0) = 0\) by def'n of integral; iv) \(E'(0) = 1\) by def'n of zero exponent

b) Explain i) why \(E(1) < 1\), and ii) why \(E(-1) < 0\).

Ans. Use part a): i) by MVT, \(E(1) - E(0) = E'(c)(1 - 0)\) so \(E(1) = E'(c)\), for some \(0 < c < 1\); but \(E'(0) = 1\) and \(E''(c) < E'(c) < E'(0) = 1\), giving \(E(1) < 1\); ii) the integrand is even so the integral is odd, whence \(E(-1) = -E(1) < 0\).

15. A pyramid of height \(h\), with square base of side \(b\), lies along the \(x\)-axis as shown.

a) Explain why the cross section at \(x\), for \(0 \leq x \leq h\), is a square of side \(bx/h\).

Ans. The cross section at \(x\) determines a pyramid of height \(x\), similar to the given one; then the base is square and the proportion of side to side is equal to that of height to height, whence side \(b = x = h\).

b) Compute the volume of the pyramid as an integral.

Ans. A cross section volume element has \(dV = (bx/h)^2 \, dx\) for \(0 \leq x \leq h\), so volume \(V = \int_0^h (bx/h)^2 \, dx = \left(\frac{b}{h}\right)^2 \int_0^h x^2 \, dx = \frac{1}{3}b^2h\)

16. a) State the Fundamental Theorem of Calculus, including all hypotheses.

Ans. Theorem. Suppose \ldots \(\text{etc.}\)

b) An animal population is increasing at a rate of \(200 + 50t\) per year \(t\) years. By how much does the animal population increase between the fourth and tenth years?

Ans. Set \(N(t) = \text{population size at time } t\). Then \(N(10) - N(4) = \int_4^{10} N'(t) \, dt = \int_4^{10} 200 + 50t \, dt = 3300\).

17. A hemispherical bowl of radius 8 in. contains punch; the punch is \(P\) in. deep. Find the volume of punch in the bowl, as follows:

a) Set up an integral for the volume, showing clearly your choice of coordinates, and indicate the element \(dV\) of volume.

b) Compute the volume of the punch.

Ans. With vertical \(h\)-axis, origin at bowl bottom, volume element is a horizontal disk, \(\pi \cdot (8 - h)^2 \, dh\), \(V = \int_0^P dV = \pi \cdot P^2 (8 - P/3)\).