**MATH 140 SAMPLE FINAL EXAMINATION**

Show all work, explain each step.

**MATH 140 CALCULUS I**

No books, no notes.

**Do twelve of the thirteen problems.**

1. **a)** Evaluate the limits. Explain your answers.
   
   \[
   \begin{align*}
   i) \lim_{x \to -5} & \frac{x^2 - 25}{x - 5} & ii) \lim_{x \to 5} & \frac{x^2 - 25}{x - 5} & iii) \lim_{x \to \infty} \frac{3x^2 + 2x}{5x^4 + 3/x} \\
   iv) \lim_{\theta \to 0} & \frac{\theta^2}{\sin^2 \theta} & v) \lim_{t \to \infty} & \frac{e^t - e^{2t}}{e^{2t} + 1} 
   \end{align*}
   \]
   
   Ans. \( i) \ 10; \\
   \( ii) \ 0; \\
   \( iii) \ \text{infinite} \\
   \( iv) \ 1; \\
   \( v) \ -1 \)

   **b)** Draw a graph illustrating your answers to \( i \) and \( ii \) above.
   
   Ans. \( y = x + 5 \) with hole at \((5, 10)\)

2. **a)** State the definition of \( f'(x) \) as a limit. Illustrate the definition with an appropriate sketch; label the elements of the sketch corresponding to the symbols in your definition.
   
   **Note.** In this problem, you must use the definition, not derivative formulae.
   
   Ans. (Sketch to illustrate that a tangent line slope is limit of secant line slopes)

   **b)** Suppose function \( F \) has the values in this table.
   
   \[
   \begin{array}{c|cccc}
   x & 0.90 & 0.99 & 1 & 1.01 \\
   \hline
   F(x) & 2.39 & 2.94 & 3 & 3.06 \\
   \end{array}
   \]
   
   Using the definition, estimate \( F'(1) \) from the table. Explain your result.
   
   Ans. \( F'(1) \approx \frac{F(1.01) - F(1)}{1.01 - 1} = \frac{-1}{-1} = 1 \), (the other fractions) \( \cdots = 6 \), so \( F'(1) \) would be 6.

   **c)** Use the definition to find \( G'(x) \) if \( G(x) = x^2 - x^3 \).
   
   **Note.** In this problem, you must use the definition, not derivative formulae.
   
   Ans. \( G'(x) = 1 - 2x \)

3. **a)** Find the first derivative of each function. Simplify your answer suitably.
   
   **Note.** Now use derivative formulae.

   \[
   \begin{align*}
   i) \ y &= xe^{x^-1} & ii) \ y &= \ln |\cos(3\theta)| \\
   iii) \ y &= \frac{4 + x^3}{x + 1} & iv) \ y &= \frac{1}{\sqrt{x^2 + x^4}} (a \text{ is a constant}) \\
   \end{align*}
   \]
   
   Ans. \( i) \ e^{x-1}(1 - x^{-1}) \); \( ii) \ -3 \tan 3\theta \); \( iii) \ (2x^3 + 3x^2 - 4)(x + 1)^{-2} \); \( iv) \ -x(a^2 + x^2)^{-3/2} \)

   **b)** \( i) \) Find the second derivative of \( y = e^{-t} \sin t \).
   
   Ans. \( y' = -y + e^{-t} \cos t \), so \( y'' = -2e^{-t} \cos t \)

   **ii)\)** For \( y = e^{-t} \sin t \), is it true that \( y'' + 2y' + 2y = 0 \)? Show your calculation.
   
   Ans. Yes

4. The area of an equilateral triangle is increasing at the rate of 2 in\(^2\) per minute. How fast is the length of a side increasing when the sides are each 6 in long?

   Ans. Area = \( S^2 \sqrt{3}/4 \), so \( S = 2\sqrt{3}/9 \) in/min

5. Write the equation of the line tangent to the graph of \( xy^2 + x^3 + y = 1 \) at the point \( P(1, 0) \).

   **Note.** Line thru \( P \), slope \( y' = -(y^2 + 3x^2)/(3xy^2 + 1) \); equation is \( y = -3(x - 1) \)

6. Sketch a graph of a function \( f \) that satisfies all of the given conditions. Show on your graph its \( i) \) symmetry, \( ii) \) intercepts, \( iii) \) extremes, \( iv) \) inflection points, \( v) \) asymptotes, and explain how you obtained each item.

   \[
   \begin{align*}
   & f(1) = 0, f(2) = 2, f(3) = 1, f(-x) = -f(x) \text{ for all } x; \\
   & x = 0 \text{ defines a vertical asymptote, } y = 0 \text{ defines a horizontal asymptote; } \\
   & f'(x) < 0 \text{ for } x < -2 \text{ and for } x > 2, f'(x) > 0 \text{ for } -2 < x < 0 \text{ and } 0 < x < 2; \\
   & f''(x) < 0 \text{ for } x < -3 \text{ and } 0 < x < 3, f''(x) > 0 \text{ for } -3 < x < 0 \text{ and } x > 3. \\
   & \text{Ans. origin sym; ints } (\pm 1, 0); \text{ loc max } (2, 2), \text{ loc min } (-2, -2); \text{ inflection points } (\pm 3, \pm 1); \text{ asy } x \text{ axes}
   \end{align*}
   \]

The exam continues on the next side.
7. A portion of a field is to be fenced off, so that the enclosed region will be a rectangle 1000 square yards in area. The fencing to be used along one side costs $10 per yard, and the fencing for the other three sides costs $6 per yard. Find the dimensions requiring the least cost of fence. Explain why the cost is least for the dimensions you find.

\[ \text{Ans. For length } l = 36.51, dC/dl = 0 \text{ and } d^2C/dl^2 > 0, \text{ then width } w = 27.39 \text{ ( } = \frac{L}{4} \text{ - length) } \]

8. a) State the Mean Value Theorem, including the hypotheses. Illustrate the Theorem with an suitable sketch; label the elements of the sketch corresponding to the symbols in the Theorem.

\[ \text{Ans. (sketch to illustrate that slope of chord equals slope of appropriate tangent) } \]

b) Use the Mean Value Theorem to show that if \( b \) is a positive number, then \( e^b - e^0 = e^c(b - 0) \) and \( b > c > 0 \); then \( e^b - e^0 = e^c b \) with \( e^c > 1 \), hence \( e^b - e^0 > b \). \( \square \)

9. A brick on a spring bobs up and down with velocity \( v(t) = 3 \sin t \) at time \( t \). The initial position of the brick is \( y(0) = -1 \).

a) i) Find the acceleration \( a(t) \) of the brick,

ii) and find the position function \( y(t) \) of the brick.

\[ \text{Ans. i) } a(t) = v(t) = 3 \cos t; \text{ ii) } y(t) = y(0) + \int_0^t v(u) \, du = 2 - 3 \cos t \]

b) How high up along the \( y \)-axis does the brick go? Explain.

\[ \text{Ans. max of } -3 \cos t = 3, \text{ so max } y = 5 \]

10. a) Write a Riemann Sum \( S \) with 3 subdivisions which approximates the area \( A \) under the graph of \( f(x) = \frac{1}{x} \) from 2 to 3.

\[ \text{Ans. a regular partition with } x_i^* = x_i \text{ (for instance) gives } A \approx S = 0.3790 \]

b) Compute the exact value \( A \) of the area.

\[ \text{Ans. } A = \ln(3/2) = 0.4055 \]

c) Find the error ratio \( \frac{|A - S|}{A} \) in your approximation \( A \approx S \).

\[ \text{Ans. } |A - S|/A = 0.065, \text{ so error } |A - S| \text{ for } S \text{ in part a) is 6.5\% of } A \]

11. A function \( F \) is defined by \( F(x) = \int_0^x \cos(t^2) \, dt \).

a) Find:

i) \( F'(x) \), ii) \( F''(x) \), iii) \( F(0) \), iv) \( F'(0) \).

\[ \text{Ans. i) } \cos(x^2); \text{ ii) } -2x \sin(x^2); \text{ iii) } 0; \text{ iv) } 1 \]

b) Find a value \( x \) where \( F \) achieves a local maximum. Explain why the value is maximum for the \( x \) you find.

\[ \text{Ans. for } x = \sqrt{\pi/2}, F'(x) = 0 \text{ and } F''(x) < 0, \text{ max } \]

12. For the integral \( \int_0^1 x^p \, dx \) with \( p > 0 \):

a) evaluate it using the Fundamental Theorem of Calculus; then

b) express the integral as a limit of Riemann sums, using right endpoints.

\[ \text{Ans. } \int_0^1 x^p \, dx = \frac{1}{p+1} = \lim_{n \to \infty} \frac{\sum_{i=1}^n (0 + \frac{i}{n})^p}{n} = \lim_{n \to \infty} \sum_{i=1}^n \frac{i^p}{n^{p+1}}. \]

c) Prove that \( \lim_{n \to \infty} \sum_{i=1}^n \frac{i^p}{n^p} = \frac{1}{p+1} \).

\[ \text{Ans. The limit is the integral with } p = 5; \text{ by FTC the integral } = \frac{1}{6}. \]

13. a) Compute the area of the region bounded below by the graph of \( y = x^2 - 2 \), and above by the graph of \( y = x \). Sketch the region.

\[ \text{Ans. area element } dA = (x - (x^2 - 2)) \, dx; \text{ area } = \int_1^2 \, dA = 4.5 \]

b) The region between the \( x \)-axis, the graph of \( y = x^2 + 1 \), and the lines \( x = -1 \) and \( x = 1 \) is revolved about the \( x \)-axis. Compute the volume of the resulting solid. Sketch the solid and show a typical cross-section (volume element).

\[ \text{Ans. vol element } dV = \pi y^2 \, dx; \text{ vol } = \int_{-1}^1 \, dV = 56\pi/15 = 11.37 \]

End of Sample Final Exam. ! The Exam may include problems which are not exactly like any of the problems here.