From a small number of rather obvious axioms, you will derive all the other properties you will need. In this section you will concentrate on the axioms that apply to the operations with numbers such as + and \(\times\). In Section 1-7 you will find the axioms that apply to the relationships between numbers, such as = and <.

**Objective:**
Given the name of an axiom that applies to + or \(\times\), give an example that shows you understand the meaning of the axiom; and vice versa.

There are eleven axioms that apply to adding and multiplying real numbers. These are called the Field Axioms, and are listed in the following table. If you already feel familiar with these axioms, you may go right to the problems in Exercise 1-2. If not, then read on!

**THE FIELD AXIOMS**

**CLOSURE**
\{real numbers\} is *closed* under addition and under multiplication.
That is, if \(x\) and \(y\) are real numbers, then

\[ x + y \text{ is a unique, real number,} \]
\[ xy \text{ is a unique, real number.} \]

**COMMUTATIVITY**
Addition and multiplication of real numbers are *commutative* operations. That is, if \(x\) and \(y\) are real numbers, then

\[ x + y \text{ and } y + x \text{ are equal to each other,} \]
\[ xy \text{ and } yx \text{ are equal to each other.} \]

**ASSOCIATIVITY**
Addition and multiplication of real numbers are *associative* operations. That is, if \(x, y,\) and \(z\) are real numbers, then

\[ (x + y) + z \text{ and } x + (y + z) \text{ are equal to each other.} \]
\[ (xy)z \text{ and } x(yz) \text{ are equal to each other.} \]

**DISTRIBUTIVITY**
Multiplication *distributes* over addition. That is, if \(x, y,\) and \(z\) are real numbers, then

\[ x(y + z) \text{ and } xy + xz \text{ are equal to each other.} \]

**IDENTITY ELEMENTS**
\{real numbers\} contains:

A *unique* identity element for *addition*, namely 0. (Because \(x + 0 = x\) for any real number \(x\).)

A *unique* identity element for *multiplication*, namely 1. (Because \(x \cdot 1 = x\) for any real number \(x\).)
INVERSES

\{\text{real numbers}\} \text{ contains:}

A unique additive inverse for every real number \( x \). (Meaning that every real number \( x \) has a real number \( -x \) such that \( x + (-x) = 0 \).)

A unique multiplicative inverse for every real number \( x \) except zero.
(Meaning that every non-zero number \( x \) has a real number \( \frac{1}{x} \) such that \( x \cdot \frac{1}{x} = 1 \).)

Notes:

1. Any set that obeys all eleven of these axioms is a field.
2. The eleven Field Axioms come in 5 pairs, one of each pair being for addition and the other for multiplication. The Distributive Axiom expresses a relationship between these two operations.
3. The properties \( x + 0 = x \) and \( x \cdot 1 = x \) are sometimes called the “Addition Property of 0” and the “Multiplication Property of 1,” respectively, for obvious reasons.
4. The number \( -x \) is called, “the opposite of \( x \),” “the additive inverse of \( x \),” or “negative \( x \).”
5. The number \( \frac{1}{x} \) is called the “multiplicative inverse of \( x \),” or the “reciprocal of \( x \).”

Closure — By saying that a set is “closed” under an operation, you mean that you cannot get an answer that is out of the set by performing that operation on numbers in the set. For example, \{0, 1\} is closed under multiplication because \( 0 \times 0 = 0 \), \( 0 \times 1 = 0 \), \( 1 \times 0 = 0 \), and \( 1 \times 1 = 1 \). All the answers are unique, and are in the given set. This set is not closed under addition because \( 1 + 1 = 2 \), and 2 is not in the set. It is not closed under the operation “taking the square root” since there are two different square roots of 1: \( +1 \) and \( -1 \).

Commutativity — The word “commute” comes from the Latin word “commutare,” which means “to exchange.” People who travel back and forth between home and work are called “commuters” because they regularly exchange positions. The fact that addition and multiplication are commutative operations is somewhat unusual. Many operations such as subtraction and exponentiation (raising to powers) are not commutative. For example,

\[ 2 - 5 \text{ does not equal } 5 - 2, \]

and

\[ 2^3 \text{ does not equal } 3^2. \]
Indeed, most operations in the real world are not commutative. Putting on your shoes and socks (in that order) produces a far different result from putting on your socks and shoes!

**Associativity**—You can remember what this axiom states by remembering that to “associate” means to “group.” Addition and multiplication are associative, as shown by

\[(2 + 3) + 4 = 9 \quad \text{and} \quad 2 + (3 + 4) = 9.\]

But subtraction is not associative. For example,

\[(2 - 3) - 4 = -5 \quad \text{and} \quad 2 - (3 - 4) = 3.\]

**Distributivity**—Parentheses in an expression such as \(2 \times (3 + 4)\) mean, “Do what is inside first.” But you don’t have to do \(3 + 4\) first. You could “distribute” a 2 to each term inside the parentheses, getting \(2 \times 3 + 2 \times 4\). The Distributive Axiom expresses the fact that you get the same answer either way. That is,

\[2 \times (3 + 4) = 14 \quad \text{and} \quad 2 \times 3 + 2 \times 4 = 14.\]

Note that multiplication does not distribute over multiplication. For example,

\[2 \times (3 \times 4) \quad \text{does not equal} \quad 2 \times 3 \times 2 \times 4,\]

as you can easily check by doing the arithmetic.

**Identity Elements**—The numbers 0 and 1 are called “identity elements” for adding and multiplying, respectively, since a number comes out “identical” if you add 0 or multiply by 1. For example,

\[5 + 0 = 5 \quad \text{and} \quad 5 \times 1 = 5.\]

**Inverses**—A number is said to be an inverse of another number for a certain operation if it “undoes” (or inverts) what the other number did. For example, \(\frac{1}{3}\) is the multiplicative inverse of 3. If you start with 5 and multiply by 3 you get

\[5 \times 3 = 15.\]

Multiplying the answer, 15, by \(\frac{1}{3}\) gives

\[15 \times \frac{1}{3} = 5,\]

which “undoes” or “inverts” the multiplication by 3. It is easy to tell if two numbers are multiplicative inverses of each other because their product is always equal to 1, the multiplicative identity element. For example,

\[3 \times \frac{1}{3} = 1.\]
Similarly, two numbers are *additive inverses* of each other if adding them to each other gives 0, the additive identity element. For example, $\frac{3}{5}$ and $-\frac{3}{5}$ are additive inverses of each other because

$$\frac{3}{5} + (-\frac{3}{5}) = 0.$$ 

The following exercise is designed to familiarize you with the names and meanings of the Field Axioms.

**EXERCISE 1-2**

*Do These Quickly*

The following problems are intended to refresh your skills. Some problems come from the last section, and others probe your general knowledge of mathematics. You should be able to do all 10 in less than 5 minutes.

1. Simplify: $11 - 3 + 5$
2. Multiply and simplify: $\left(\frac{2}{3}\right)\left(\frac{6}{7}\right)$
3. Add: $3.74 + 5$
4. If $x + 7$ is 42, what does $x$ equal?
5. Is $-13$ an integer?
6. Multiply: $(9x)(6x)$
7. Square 7.
8. Is 1.3 a rational number?
9. Multiply: $5(3x - 8)$
10. Simplify: $(-3)(0.7)(-5)(-1)$

Work the following problems.

1. Tell what is meant by
   a. additive identity element,
   b. multiplicative identity element.
2. What is
   a. the *additive* inverse of $\frac{3}{5}$?
   b. the *multiplicative* inverse of $\frac{3}{5}$?
3. Using variables ($x$, $y$, $z$, etc.) to stand for numbers, write an example of each of the eleven field axioms. Try to do this by writing all eleven
examples first, then checking to be sure you are right. Correct any
which you left out or got wrong.

4. Explain why 0 has no multiplicative inverse.

5. The Closure Axiom states that you get a unique answer when you add
two real numbers. What is meant by a "unique" answer?

6. You get the same answer when you add a column of numbers "up" as
you do when you add it "down." What axiom(s) show that this is
true?

7. Calvin Butterball and Phoebe Small use the distributive property as
follows:

Calvin: \(3(x + 4)(x + 7) = (3x + 12)(x + 7)\).
Phoebe: \(3(x + 4)(x + 7) = (3x + 12)(3x + 21)\).

Who is right? What mistake did the other one make?

8. Write an example which shows that:
   a. Subtraction is not a commutative operation.
   b. {negative numbers} is not closed under multiplication.
   c. {digits} is not closed under addition.
   d. {real numbers} is not closed under the \(\sqrt{}\) operation (taking the
      square root).
   e. Exponentiation ("raising to powers") is not an associative opera-
      tion. (Try \(4^{2^3}\).)

9. For each of the following, tell which of the Field Axioms was used,
   and whether it was an axiom for addition or for multiplication. Assume
   that \(x\), \(y\), and \(z\) stand for real numbers.
   a. \(x + (y + z) = (x + y) + z\)
   b. \(x \cdot (y + z)\) is a real number
   c. \(x \cdot (y + z) = x \cdot (z + y)\)
   d. \(x \cdot (y + z) = (y + z) \cdot x\)
   e. \(x \cdot (y + z) = xy + xz\)
   f. \(x \cdot (y + z) = x \cdot (y + z) + 0\)
   g. \(x \cdot (y + z) + ([x \cdot (y + z)]) = 0\)
   h. \(x \cdot (y + z) = x \cdot (y + z) \cdot 1\)
   i. \(x \cdot (y + z) \cdot \frac{1}{x \cdot (y + z)} = 1\)

10. Tell whether or not the following sets are fields under the opera-
tions + and \(\times\). If the set is not a field, tell which one(s) of the Field
    Axioms do not apply.
   a. {rational numbers}
   b. {integers}
   c. {positive numbers}
   d. {non-negative numbers}
Q7. Write the multiplicative identity element.
Q8. If $3x$ equals 42, what does $x$ equal?
Q9. Multiply: $(2.3)(4)$
Q10. Divide and simplify: $(\frac{3}{5}) ÷ (\frac{7}{9})$

For Problems 1 through 10, carry out the indicated operations in the agreed-upon order.

1. $5 + 6 \times 7$
2. $3 + 8 \times 7$
3. $9 - 4 + 5$
4. $11 - 6 + 4$
5. $12 ÷ 3 \times 2$
6. $18 ÷ 9 \times 2$
7. $7 - 8 ÷ 2 + 4$
8. $24 - 12 \times 2 + 4$
9. $16 - 4 + 12 ÷ 6 \times 2$
10. $50 - 30 \times 2 + 8 ÷ 2$

For Problems 11 through 24, evaluate the given expression

(a) for $x = 2$
(b) for $x = -3$.

11. $4x - 1$
12. $3x - 5$
13. $|3x - 5|$
14. $|4x - 1|$
15. $5 - 7x - 8$
16. $8 - 5x - 2$
17. $|8 - 5x| - 2$
18. $|5 - 7x| - 8$
19. $x^2 - 4x + 6$
20. $x^2 + 6x - 9$
21. $4x^2 - 5x - 11$
22. $5x^2 - 7x + 1$
23. $5 - 2 \cdot x$
24. $3 + 4 \cdot x$

For Problems 25 through 40, simplify the given expression.

25. $6 - [5 - (3 - x)]$
26. $2x - [3x + (x - 2)]$
27. $7(x - 2(3 - x))$
28. $3(6x - 5(x - 1))$
29. $7 - 2[3 - 2(x + 4)]$
30. $8 + 4[5 - 6(x - 2)]$
31. $3x - [2x + (x - 5)]$
32. $4x - [3x - (2x - x)]$
33. $6 - 2[x - 3 - (x + 4) + 3(x - 2)]$
34. $7[2 - 3(x - 4) + 4(x - 6)]$
35. $6[x - \frac{1}{3}(x - 1)]$
36. $8[2x - \frac{1}{2}(6x + 5)]$