Proof of the Extreme Value Theorem.

Theorem (Extreme Value Theorem): If \( f(x) \) is a function defined and continuous on the closed interval \([a,b]\), then \( f \) attains its maximum at some point \( c \in [a,b] \).

Outline of proof:

0. **Definition.** An infinite sequence \(<x_n| n \in \mathbb{N}> \) converges to \( x_0 \) iff \( \forall \varepsilon > 0, \exists N \in \mathbb{N} \) such that \( \forall n \in \mathbb{N}, |x_n - x_0| < \varepsilon \).

1. **Bolzano-Weierstrass Theorem.** Every bounded infinite sequence has a convergent subsequence.

2. The range of a continuous function \( f \) with domain the closed interval \([a,b]\) is bounded. That is, \( \exists N \in \mathbb{N}, \forall x \in [a,b], f(x) < N \).

3. There is a least upper bound for the range of any continuous function defined on a closed interval.

4. The function in #2,3 actually attains its maximum, for some \( c \in [a,b] \).

Proof:

**Step 1.** Bolzano-Weierstrass Theorem. Every bounded infinite sequence has a convergent subsequence.

We may assume that \([a,b] = [0,1]\).

Suppose \(<x_n| n \in \mathbb{N}> \) is a sequence of elements on the interval \([0,1]\).

Define a sequence of intervals \( S_i \) and points \( M_i \) and \( y_i \) as follows:

Let \( S_1 = [0,1] \). Let \( M_1 = \) the midpoint of \( S_1 \). \( M_1 = \frac{1}{2} \). Let \( y_1 = x_1 \).

Then either \([0, M_1]\) or \([M_1, 1]\) or both, must contain an infinite number of the \( x_i \)'s, otherwise the sequence could not be an infinite sequence.

If \([0, M_1]\) contains an infinite number of the \( x_i \)'s, let \( S_2 = [0, M_1]\).

Otherwise, let \( S_2 = [M_1, 1] \).

Let \( y_2 \) = the first \( x_i \) beyond \( x_1 \) in the sequence \(<x_n>\) for which \( y_2 \in S_2 \).

Continue this process infinitely, choosing if possible the left-hand of the two intervals at each step. Then, \(<y_n| n \in \mathbb{N}> \) is a subsequence of \(<x_n>\).

From the sequence of left endpoints of \( S_n \), define the number \( c \) with a binary expansion

with the \( n-1^{th} \) binary place \( = 0 \) or \( 1 \), according as \( S_n \) was the left or right interval chosen.

Claim: the sequence \(<y_n| n \in \mathbb{N}> \) converges to the point \( c \).

The claim is true because each interval \( S_{n+1} \) is contained in the interval \( S_n \), and the width of \( S_n = (1/2^n) \). Therefore the sequence \(<y_n>\) is a convergent subsequence of \(<x_n>\).

**Step 2.** The range of a continuous function \( f \) with domain the closed interval \([a,b]\) is bounded. That is, \( \exists N \in \mathbb{N}, \forall x \in [a,b], f(x) < N \).

Proof: Consider the sequence of sets \( T_n = \{x \in [a,b] | f(x) > n\} \), where \( n \in \mathbb{N} \).

Each \( T_n \) contains \( T_{n+1} \).
Proof of the Extreme Value Theorem.

Either all of the $T_n$ are not empty, or there is an $N$ such that $\forall n > N, T_n$ is empty.

If all $T_n$ are not empty, then let $x_n = \text{the midpoint of } T_n$.

The sequence $<x_n>$ is an infinite sequence in a bounded closed interval.

By Step (1), the Bolzano-Weierstrass theorem, there exists a convergent subsequence $<y_n>$ of $<x_n>$. Suppose $<y_n>$ converges to $c$.

Since $f$ is a continuous function, the sequence $<f(y_n)>$ must converge to $f(c)$.

But $f(y_n) > n$. So the limit as $n$ approaches infinity of $f(y_n) = \infty$. This means $f(c)$ is not a finite number, which means $f$ is undefined at $c$, contradicting the assumption that $f$ is defined at every number in $[a, b]$.

Therefore, some $T_n$ is empty. Therefore that number $n$ is an upper bound of the range of $f$.

Step 3. The range of the function $f$, in step 2 above, has a least upper bound.

Proof: since the range of $f$ is a subset of the Real Numbers, and it is bounded above, then the range has a least upper bound, by the Completeness property of the Real Numbers.

Step 4. The function in #2,3 actually attains its maximum, for some $c \in [a, b]$.

Proof:

Let $L = \text{the upper bound of the range of } f$. Let $U_n = \{x \in [a, b] \mid f(x) > L - 1/n\}$.

If for some $n$, $U_n$ were empty, then $L$ could not be the least upper bound of $f(x)$, because $L - 1/n$ would be a smaller upper bound. Therefore, all $U_n$ are non-empty.

For each $n$, let $q_n$ be an element of $U_n$. Then, the sequence $<q_n>$ is an infinite sequence, and, by the Bolzano-Weierstrass Theorem, has a convergent subsequence $<r_n>$. For each $n$, $f(r_n) \geq f(q_n)$. Let $c$ be the limit of the sequence $<r_n>$. Since $f(r_n) \geq f(q_n) > L - 1/n$, the limit as $n \to \infty$ of $f(q_n)$ must not be less than $L$. Hence, since $f$ is continuous, $f(c) = L$, and $f$ attains its maximum at $c$.

End of proof.