# Unitary representations of dihedral graded Hecke algebras

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# Outline

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5. Intertwining operators and $\rho(A(\nu))$
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   - $\mathbb{H}(I_2(2m+1))$
   - $\mathbb{H}(H_3)$
   - $\mathbb{H}(I_2(2m))$

# Context

$G$ split, connected, linear algebraic reductive group over $p$-adic $\mathbb{F}$.
Choose $\mathcal{I}$, an Iwahori subgroup of $G$.
Affine Iwahori-Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{I} \backslash G / \mathcal{I})$ under convolution has $*$ operation $f^*(x) = \overline{f(x^{-1})}$.

## Theorem (Borel, 1976, and Casselman)

*$V \to V^{\mathcal{I}}$ defines a category equivalence*
*$C(\mathcal{I}) = \{$Iwahori-spherical reps$\} \leftrightarrow C(\mathcal{H}) = \{$finite dim reps of $\mathcal{H}\}$*

## Theorem (Barbasch-Moy, 1989, 1993)

*$V \in C(\mathcal{I})$ is unitary if and only if $V^{\mathcal{I}} \in C(\mathcal{H})$ is unitary.*

## Theorem (Lusztig, 1989)

*There is a filtration on $\mathcal{H}$ giving rise to a graded Hecke algebra, $\mathbb{H}$.*
*Isomorphism between completions relates representations.*

# Context

Full unitary dual known for $p$-adic groups $\mathrm{GL}(n)$ [Tadic, 1986], $G_2$ [Muic, 1997]

Spherical unitary dual has been determined for

- classical $p$-adic groups (types $B$, $C$, and $D$) via their graded Hecke algebra [Barbasch-Moy, 1996]
- split real and $p$-adic groups of type $\mathrm{Sp}(n)$ and $\mathrm{SO}(n)$ [Barbasch, 2010]
- graded Hecke algebras of type $B/C$ with unequal parameters [Barbasch-Ciubotaru, 2005]
- split $p$-adic groups of type $F_4$ and type $E$ (generic dual) [Ciubotaru, 2005; Barbasch-Ciubotaru, 2009] and
- geometric graded Hecke algebras with unequal parameters [Ciubotaru, 2008]

I-M involution $\Rightarrow$ determines Whittaker-generic dual.

# Noncrystallographic real reflection groups

$I_2(n)$ = dihedral group of $2n$ symmetries of the regular $n$-gon

$= \text{Coxeter group } \langle s_1, s_2 \mid s_i^2 = (s_1 s_2)^n = 1 \rangle \qquad \circ_1 \overset{n}{\text{———}} \circ_2$

$H_3$ = 120 symmetries of the icosahedron or dodecahedron

$= \text{Coxeter group } \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^5 = (s_2 s_3)^3 = (s_1 s_3)^2 = 1 \rangle$

with diagram $\circ_1 \overset{5}{\text{———}} \circ_2 \text{———} \circ_3$

$\simeq Alt_5 \times \mathbb{Z}_2$

$H_4$ = 14,400 symmetries of the 120-cell or 600-cell

$= \langle s_1, s_2, s_3, s_4 \mid \text{same plus } s_4^2 = (s_3 s_4)^3 = (s_1 s_4)^2 = (s_2 s_4)^2 = 1 \rangle$

with diagram $\circ_1 \overset{5}{\text{———}} \circ_2 \text{———} \circ_3 \text{———} \circ_4$

# Why study noncrystallographic cases?

- No associated Lie group, $p$-adic group or geometry, but representation theory of noncrystallographic $\mathbb{H}$ is both similar and intriguingly different from crystallographic cases.
  1. nonspherical tempered rep in $\mathbb{H}(H_3)$ [K, 1999], reps semisimple for abelian subalgebra [K-Ram, 2002]
  2. solvable points and spin representations [Chan, 2012]
- Algebras related to complex reflection groups. [Ram-Shepler, 2003], [Etingof-Ginzburg, 2002], [Chmutova, 2006], [Balagović-Puranik, 2010], [Etingof-Stoica,Griffeth, 2009]
- Other mysterious considerations.
  1. associahedra and noncrossing and nonnesting partitions [Armstrong, 2009], [Brady-Watt, 2008], [Chen-K, 2007]
  2. critical pts of function on manifold, sing bdry [Lyashko, 1982], wavefront singularities from obstacles [Scherbaks, 1982-2003]
  3. does $\exists$ topological object with moment graph as linear graph from noncrystallographic reflection group? [MacPherson, 2004]

# Root system and associated hyperplanes

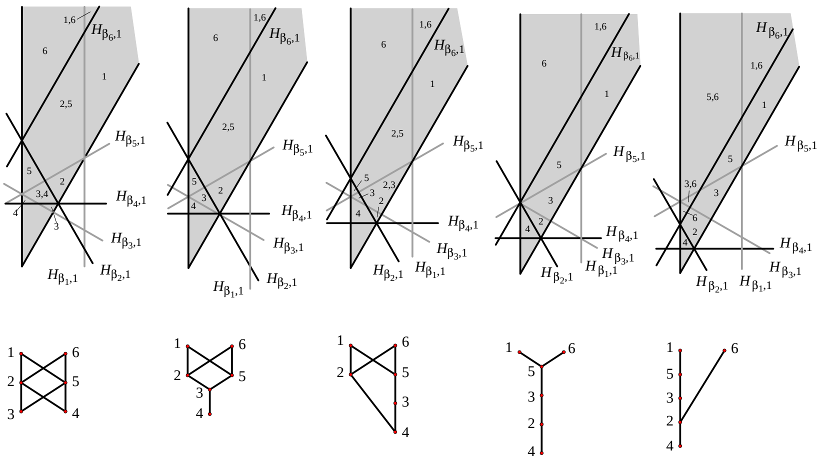
Root system $(\mathfrak{h}_{\mathbb{R}}^*, R, \mathfrak{h}_{\mathbb{R}}, R^\vee)$ not assumed to be crystallographic.
Choice of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$
Fix $c : R \to \mathbb{R}$ with $c_\alpha = c_\beta$ if $\beta = \alpha$ and write $c(\alpha_i)$ as $c_i$.

- $H_{\alpha_i}^0 = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid x(\alpha_i^\vee) = 0\}$, $1 \leq i \leq n$ (reflecting)
- $H_{\alpha_i}^{c_i} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid x(\alpha_i^\vee) = c_i\}$, $1 \leq i \leq n$ (affine)
- $\overline{C} =$ closure of fundamental chamber
- $\overline{C}$ is divided into cells of dimension $0, 1, \dots, n$ by $\{H_{\alpha_i}^{c_i}\}_{i=1}^n$
- $Z(x) = \{\alpha_i \mid x(\alpha_i^\vee) = 0\}$ - which reflecting planes contain $x$,
- $P(x) = \{\alpha_i \mid x(\alpha_i^\vee) = c_i\}$ - which affine planes contain $x$
- can use $Z(x)$ and $P(x)$ to characterize the 0-dim cells.

### Theorem (Chen-K, 2007)

*Ideals (equivalently antichains) in poset of positive roots under the root order characterize 2-dim cells for $I_2(n)$ and 3-dim cells for $H_3$.*

# Dominant regions and positive root order posets for $I_2(6)$




$\frac{\sqrt{3}}{2} < r < \frac{2}{\sqrt{3}}$ $(r = 1)$ $r = \frac{2}{\sqrt{3}}$ $\frac{2}{\sqrt{3}} < r < \sqrt{3}$ $r = \sqrt{3}$ $\sqrt{3} < r$

# Graded Hecke algebra $\mathbb{H}$

- $W$ acts on complexification $\mathfrak{h}_{\mathbb{C}}^* = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}^*$.
- $W$ acts on symmetric algebra $S(\mathfrak{h}_{\mathbb{C}}^*) \cong$ polynomials on $\mathfrak{h}_{\mathbb{C}}$.
- $\mathbb{C}W = \left\{ \sum_{w \in W} a_w t_w \mid a_w \in \mathbb{C} \right\}$ with multiplication as in $W$.
- Fix indeterminate $r$ and recall $c : R \longrightarrow \mathbb{R}$.

Graded Hecke algebra is $\mathbb{H} = \mathbb{C}[r] \otimes \mathbb{C}W \otimes S(\mathfrak{h}_{\mathbb{C}}^*)$ with multiplication as in $S(\mathfrak{h}_{\mathbb{C}}^*)$ and in $\mathbb{C}W$ and

$$x t_{s_i} = t_{s_i} s_i(x) + r c_i x(\alpha_i^\vee) \qquad \text{for } x \in \mathfrak{h}_{\mathbb{C}}^*.$$

Specialize $r = 1$.

# Tempered representations of $\mathbb{H}$

Center $Z(\mathbb{H}) = S(\mathfrak{h}_{\mathbb{C}}^*)^W = W$-invariant polynomials on $\mathfrak{h}_{\mathbb{C}}$.
$Z(\mathbb{H})$ acts on irr $\mathbb{H}$-module $V$ by central character $\gamma \in \mathfrak{h}_{\mathbb{C}}$ $(W\gamma)$.
Consider only real central character $\gamma \in \mathfrak{h}_{\mathbb{R}}$.
**Goal:** Describe the real support of generic unitary reps of $\mathbb{H}$.
Any irreducible $\mathbb{H}$-module $V$ has generalized weight space decomposition

$$V = \bigoplus_{\gamma \in \mathfrak{h}_{\mathbb{C}}} V_\gamma^{\mathrm{gen}},$$

where $\gamma$ is a weight of $V$ if $V_\gamma^{\mathrm{gen}} \neq 0$.

$\mathbb{H}$-module $V$ is tempered (discrete series) if for all weights $\gamma \in \mathfrak{h}_{\mathbb{C}}$ of $V$ and all fundamental weights $\omega_i \in \mathfrak{h}_{\mathbb{C}}^*$, $\omega_i(\gamma) \leq 0$ $(\omega_i(\gamma) < 0)$.

# Unitary representations of $\mathbb{H}$

$*$-operation defined on $\mathbb{H}$ by

$$t_w^* = t_{w^{-1}}, \qquad x^* = -t_{w_0}(w_0\bar{x})t_{w_0} = -\bar{x} + \sum_{\alpha \in R^+} \bar{x}(\alpha^\vee) c_\alpha t_{s_\alpha},$$

for $w \in W$ and $x \in \mathfrak{h}_{\mathbb{C}}^*$.
$\mathbb{H}$-module $V$ is Hermitian if there is nondegenerate form such that

$$\langle h v_1, v_2 \rangle = \langle v_1, h^* v_2 \rangle, \qquad \text{for all } h \in \mathbb{H},\ v_1, v_2 \in V,$$

and $V$ is unitary if $\langle\ ,\ \rangle$ is also positive definite.
An invariant Hermitian form on $V$ is equivalent to an $\mathbb{H}$-module isomorphism from $V$ to its Hermitian dual, $V^h$.

# Langlands classification for $\mathbb{H}$

Fix subset $\Pi_M$ of simple roots and generate root system $R_M$.

$\mathbb{H}$ has subalgebra $\mathbb{H}_M = \mathbb{H}_{M_0} \otimes S(\mathfrak{t}^*) \leftrightarrow (\mathfrak{h}_{\mathbb{C}}^*, R_M)$ where

$\mathbb{H}_{M_0} \leftrightarrow (\mathbb{C}\langle \Pi_M \rangle, R_M)$, and define

$$\mathfrak{t} = \{\nu \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(\nu) = 0 \text{ for all } \alpha \in \Pi_M\}$$
$$\mathfrak{t}^+ = \{\nu \in \mathfrak{t} \mid Re(\alpha(\nu)) > 0 \text{ for all } \alpha \in \Pi \setminus \Pi_M\}.$$

For an $\mathbb{H}_M$-module $U$ let $I(M, U) = \mathbb{H} \otimes_{\mathbb{H}_M} U$.

## Theorem (Langlands Classification - Evens, 1996)

1. *Every irr $\mathbb{H}$-module is a quotient of a standard module* $X(M, \sigma, \nu) = I(M, \sigma \otimes \mathbb{C}_\nu)$, *$\sigma$ a tempered $\mathbb{H}_{M_0}$-mod, $\nu \in \mathfrak{t}^+$.*
2. *$X(M, \sigma, \nu)$ has unique irr (Langlands) quotient $L(M, \sigma, \nu)$.*
3. *$L(M, \sigma, \nu)$ is unique up to conjugacy of $(M, \sigma, \nu)$.*

# Intertwining operator generalities

Restrict to generic representations $V$, i.e., $\mathrm{Hom}_W(V, \mathrm{sgn}) \neq 0$.
Writing $w_0 = s_{i_1} \cdots s_{i_\ell}$ induces the ordering

$$s_{i_\ell} \cdots s_{i_2} \alpha_{i_1}, \quad \ldots, \quad s_{i_\ell} \alpha_{i_{\ell-1}}, \quad \alpha_{i_\ell},$$

For each $\nu \in \mathfrak{h}_{\mathbb{R}}^*$, define the following element of $\mathbb{R}[W]$:

$$A(\nu) = \left(1 + \frac{1}{c_{i_1}} \langle \nu, s_{i_\ell} \cdots s_{i_2} \alpha_{i_1}^\vee \rangle s_{i_1}\right) \cdots \left(1 + \frac{1}{c_{i_\ell}} \langle \nu, \alpha_{i_\ell}^\vee \rangle s_{i_\ell}\right)$$

- $A(\nu)$ is independent of choice of reduced expression for $w_0$.
- $A(\nu)$ is invertible if and only if $\langle \nu, \beta^\vee \rangle \neq \pm c_\beta$ for all $\beta \in R^+$.
- If $w_0 \nu = -\nu$, then $\rho(A(\nu))$ is Hermitian for all unitary $\rho \in \hat{W}$.

# Intertwining operator, cont'd.

Define the principal series module $X(\nu) = \mathbb{H} \otimes_{S(\mathfrak{h}^*_{\mathbb{C}})} \mathbb{C}_\nu$ for $\nu \in \mathfrak{h}_{\mathbb{R}}$.
If $\nu \in C$ then $X(\nu)$ has a unique irreducible quotient $L(\nu)$, which is generic, and any generic $\mathbb{H}$-module appears in this way.
$L(\nu)$ is Hermitian if and only if $w_0\nu = -\nu$.

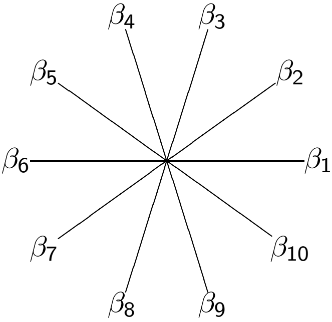
- $A(\nu)$ is an intertwining operator from $X(\nu)$ to $X(-\nu)$ with image $L(\nu)$ if $w_0\nu = -\nu$.
- $L(\nu)$ is unitary if and only if $\rho(A(\nu))$ is positive semi-definite for all $\rho \in \widehat{W}$.
- The signature of $\rho(A(\nu))$ is constant on cells.

# Root system of type $I_2(n)$

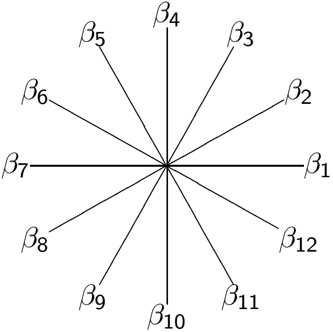
Set $\theta = \frac{\pi}{n}$, $\{\mathbf{e}_1, \mathbf{e}_2\}$ standard orthonormal basis for $\mathfrak{h}^*_{\mathbb{R}}$.

$$R := \{\, \beta_k = (\cos(k-1)\theta)\mathbf{e}_1 + (\sin(k-1)\theta)\mathbf{e}_2 \mid 1 \le k \le 2n \,\}$$

Choose $\quad \Pi := \{\, \alpha_1 = \beta_1, \alpha_2 = \beta_n \,\} \quad R^+ := \{\, \beta_k \mid 1 \le k \le n \,\}$




$I_2(5)$




$I_2(6)$

# Intertwining operator for $I_2(2m+1)$

For $W = I_2(2m+1)$, $w_0 = s_m \cdots s_2 s_1 s_2 \cdots s_m$ induces the ordering

$$\{\alpha_2 = \beta_{2m+1}, \beta_{2m}, \ldots, \beta_2, \beta_1 = \alpha_1\} \text{ if } m \text{ is even, and}$$
$$\{\alpha_1 = \beta_1, \beta_2, \ldots, \beta_{2m}, \beta_{2m+1} = \alpha_2\} \text{ if } m \text{ is odd.}$$

Since $w_0 = (s_m \cdots s_2) s_1 (s_m \cdots s_2)^{-1} = s_{(s_m \cdots s_2)(\alpha_1)} = s_{\beta_{n+1}}$,

$$\begin{aligned} V_0^+ &= \{\nu \in \mathfrak{h}_{\mathbb{R}}^* \mid w_0 \nu = -\nu\} \cap \overline{C} = \{0\} \cup \{\tfrac{1}{t}\beta_{m+1} \mid t \in \mathbb{R}^{>0}\} \\ &= \text{ points along the central ray of } C. \end{aligned}$$

For $\nu = \frac{1}{t}\beta_{m+1} \in V_0^+$, $\langle \nu, \beta_{2m+1-k}^\vee \rangle = \langle \nu, \beta_{k+1}^\vee \rangle$, hence

$$A(\nu) = \left(1 + \tfrac{1}{c_1}\langle \nu, \beta_1^\vee \rangle s_m\right) \cdots \left(1 + \tfrac{1}{c_1}\langle \nu, \beta_{m+1}^\vee \rangle s_1\right) \cdots \left(1 + \tfrac{1}{c_1}\langle \nu, \beta_1^\vee \rangle s_m\right)$$

# $\rho^h(A(\nu))$ for $I_2(2m+1)$

For $0 \leq h \leq m$, let

$$A_1^h := \rho^h(s_1) = \begin{bmatrix} \cos(h\theta) & \sin(h\theta) \\ \sin(h\theta) & -\cos(h\theta) \end{bmatrix}$$
$$A_2^h := \rho^h(s_2) = \begin{bmatrix} \cos(h\theta) & -\sin(h\theta) \\ -\sin(h\theta) & -\cos(h\theta) \end{bmatrix}.$$

Then $\rho^0 = \rho^+ \oplus \rho^-$ and $\hat{W} = \{\rho^+, \rho^-, \rho^1, \ldots, \rho^m\}$.
Since

$$\langle \beta_{m+1}, \beta_k^\vee \rangle = 2\cos((m-k+1)\theta), \text{ use } c_1 = 2 \text{ so}$$

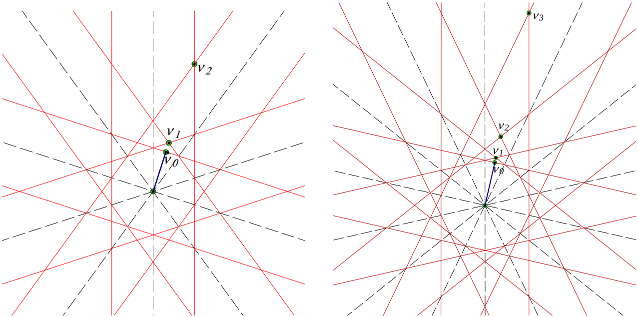
$$t^{2m+1}\rho^h\left(A\left(\tfrac{1}{t}\beta_{m+1}\right)\right) = \cdots(tI + \cos\theta A_2^h)(tI + A_1^h)(tI + \cos\theta A_2^h)\cdots$$
$$\rho^h\left(A\left(\tfrac{1}{t}\beta_{m+1}\right)\right) \text{ psd} \iff \rho^h(A(t)) := t^{2m+1}\rho^h\left(A\left(\tfrac{1}{t}\beta_{m+1}\right)\right) \text{ psd}$$

# The generic unitary dual for $\mathbb{H}(I_2(2m+1))$

## Theorem (Chen-K)

*For $\mathbb{H}(I_2(2m+1))$ the $\nu$ in $V_0^+$ that support generic unitary reps are $\nu = 0$, and $\nu = \frac{1}{t}\beta_{m+1}$ for $t \geq 1$, $t = \cos k\theta$ for $k = 1, \ldots, m$.*

Figure: Dominant support of the generic unitary dual for $I_2(5)$ and $I_2(7)$



# Proof - nondistinguished points

If $t < 1$ and $t \neq \cos k\theta$ for $1 \leq k \leq m$ then

$$\rho^{-}(A(t)) = (t-1)\prod_{k=1}^{m}(t-\cos(k\theta))^2 < 0$$

and $\nu = \frac{1}{t}\beta_{m+1}$ is nonunitary.
$\nu = 0$ is unitary since $\forall\rho$, $\rho(A(0)) = I$ is positive semidefinite.
$\nu = \frac{1}{t}\beta_{m+1}$ for $t \geq 1$ is unitary because

$$\rho^{\pm}(A(t)) = (t \pm 1)\prod_{k=1}^{m}(t \pm \cos(k\theta))^2 \geq 0$$

and $M_0 = tI + A_1^h$ with eigenvalues $t \pm 1$ is psd so for all $\rho^h$

$$\rho^h(A(t)) = (\sqrt{M_0}M_1 \dots M_m)^T(\sqrt{M_0}M_1 \dots M_m) \text{ is psd.}$$

# Proof - distinguished points

$\nu_k = \frac{1}{t}\beta_{m+1}$ for $t = \cos k\theta$, $k = 1, \dots, m$ are unitary by

Theorem (Ciubotaru-Opdam-Trapa, 2012)

*If $\mathbb{H}$ a graded Hecke algebra, $L$ a discrete series rep of $\mathbb{H}$ with $\chi =$ character of $L|_{\mathbb{C}W}$, and*

$$\langle L, L\rangle^{ell} := \sum_{i\geq 0}(-1)^i \langle \chi, \chi \otimes \wedge^i \chi_{ref}\rangle_W = 1,$$

*then $L$ is unitary.*

Apply to discrete series $L_{c,k+1}$, $0 \leq k \leq n-1$, at distinguished $\nu$, with character on $\mathbb{C}W$ of $\chi_k = \chi^+ + \sum_{j=1}^{k} \chi^j$ [K-Ram, 2002].

# Proof - distinguished points - elliptic product

In $I_2(n)$, $\chi^{\text{ref}} = \chi^1, \quad \wedge^0\chi^1 = \chi^+, \quad \wedge^1\chi^1 = \chi^1, \quad \wedge^2\chi^1 = \chi^-$,
and for $1 \leq j \leq n-1$,

$$\chi^j \otimes \chi^\pm = \chi^j \text{ and}$$
$$\chi^j \otimes \chi^1 = \chi^{j-1} + \chi^{j+1} \text{ (recall } \chi^0 = \chi^+ + \chi^-\text{).}$$

By orthogonality, with $\chi_k = \chi^+ + \sum_{j=1}^{k} \chi^j$ for $0 \leq k \leq n-1$,

$$\langle \chi_k, \chi_k \otimes \wedge^0\chi^1 \rangle = \langle \chi_k, \chi_k \rangle = k+1,$$
$$\langle \chi^k, \chi^k \otimes \wedge^1\chi^1 \rangle = \langle \chi^k, \chi^k \otimes \chi^1 \rangle = 2k,$$
$$\langle \chi^k, \chi^k \otimes \wedge^2\chi^1 \rangle = \langle \chi^k, \chi^k \otimes \chi^- \rangle = k,$$

so $\langle L_{c,k+1}, L_{c,k+1} \rangle^{\text{ell}} = k+1-2k+k = 1 \Rightarrow L_{c,k+1}$ is unitary. $\square$

# Elementary path to unitarity at distinguished points

Set $t = \cos(k\theta)$ and let

$$M_{2j} = \cos(k\theta)I + \cos(2j\theta)A_1^h$$
$$M_{2j+1} = \cos(k\theta)I + \cos((2j+1)\theta)A_2^h.$$

$\det(M_k) = 0 \Rightarrow \det(\rho^h(A(t))) = 0$ and remaining $M_j$ nonsingular.
Truncate $\rho^h(A(t))$ at $M_k$. Find nonzero $v$ in $\mathrm{Im}((M_0 M_1 \cdots) M_k)$.

## Conjecture

$$P := \mathbf{e}_1^T M_k \cdots M_2 M_1 M_0 M_1 M_2 \cdots M_k \mathbf{e}_1$$
$$= 2^{2k+1} \cos^2\left(\frac{h\theta}{2}\right) \prod_{j=1}^{k} \cos^2(j\theta) \cos\left(\left(h + \frac{2j-1}{2}\right)\theta\right) \cos\left(\left(h - \frac{2j-1}{2}\right)\theta\right)$$

Conj $\Rightarrow P \geq 0$: $\cos\left(\left(h \pm \frac{2j-1}{2}\right)\theta\right) < 0 \Rightarrow$ some $\cos\left(\left(h \pm \frac{2\ell-1}{2}\right)\theta\right) = 0$

# Unitarity at distinguished points - reduction

$$P \stackrel{?}{=} 2^{2k+1} \cos^2\left(\frac{h\theta}{2}\right) \prod_{j=1}^{k} \cos^2(j\theta) \cos\left(\left(h + \frac{2j-1}{2}\right)\theta\right) \cos\left(\left(h - \frac{2j-1}{2}\right)\theta\right)$$

If $k$ is even, $M_k \mathbf{e}_1 = 2\cos\left(\frac{h}{2}\theta\right)\cos(k\theta) v_1$.
If $k$ is odd, $M_k \mathbf{e}_1 = 2\cos\left(\frac{h}{2}\theta\right)\cos(k\theta)[\cos(h\theta) v_1 + \sin(h\theta) v_2]$.
So for $0 \le i \le k-1$, define $a_i$ and $b_i$ by

$$a_i v_1 + b_i v_2 = M_{k-i} \cdots M_k \frac{1}{2\cos\left(\frac{h}{2}\theta\right)\cos(k\theta)} \mathbf{e}_1.$$

$$M_0(a_{k-1} v_1 + b_{k-1} v_2) = 2\cos^2\left(\frac{k}{2}\theta\right) a_{k-1} v_1 - 2\sin^2\left(\frac{k}{2}\theta\right) b_{k-1} v_2,$$

$$\implies P = 2^3 \cos^2\left(\frac{h}{2}\theta\right)\cos^2(k\theta)\left[a_{k-1}^2 \cos^2\left(\frac{k}{2}\theta\right) - b_{k-1}^2 \sin^2\left(\frac{k}{2}\theta\right)\right].$$

Reduce to

$$a_{k-1}\cos\left(\frac{k}{2}\theta\right) \pm b_{k-1}\sin\left(\frac{k}{2}\theta\right) \stackrel{?}{=} 2^{k-1} \prod_{j=1}^{k-1}\cos(j\theta) \prod_{j=1}^{k} \cos\left(h \pm (-1)^{j+1}\frac{2j-1}{2}\right)\theta.$$

# Unitarity at distinguished points - recurrences

$$a_{k-1}\cos\left(\tfrac{k}{2}\theta\right) \pm b_{k-1}\sin\left(\tfrac{k}{2}\theta\right) \stackrel{?}{=} 2^{k-1}\prod_{j=1}^{k-1}\cos(j\theta)\prod_{j=1}^{k}\cos\left(h \pm (-1)^{j+1}\tfrac{2j-1}{2}\right)\theta.$$

Step 1 Find formulas for $a_{k-1}$ and $b_{k-1}$. Done.
Step 2 Reduce result to product. Done for $k \leq 4$.
Step 1 Define $a_i$ and $b_i$ by

$$a_i v_1 + b_i v_2 = M_{k-i}\cdots M_k \frac{1}{2\cos\left(\frac{h}{2}\theta\right)\cos(k\theta)}\mathbf{e}_1,$$

If $k-i$ is even,

$$a_i = [\cos(k\theta) + \cos((k-i)\theta)]a_{i-1}$$
$$b_i = [\cos(k\theta) - \cos((k-i)\theta)]b_{i-1}$$

If $k-i$ is odd,

$$a_i = [\cos(k\theta) + \cos((k-i)\theta\cos(2h\theta)]a_{i-1} - \cos((k-i)\theta)\sin(2h\theta)b_{i-1}$$
$$b_i = [\cos(k\theta) - \cos((k-i)\theta\cos(2h\theta)]b_{i-1} - \cos((k-i)\theta)\sin(2h\theta)a_{i-1}.$$

# Unitarity at distinguished points - formula for $a_i, b_i$ if $k$ odd

## Proposition

*If $k$ is odd,*

$$a_{2\ell} = \sum_{\substack{(i_1,\dots,i_n) \\ \subseteq(\ell,\dots,1)}} (-1)^n [\sin(2h\theta)]^n \Pi_{(i_1,\dots,i_n)} \begin{cases} a_0 & \text{if } n \text{ is even}, \\ b_0 & \text{if } n \text{ is odd}, \end{cases} \quad \textit{where}$$

$$\begin{aligned} \Pi_{(i_1,\dots,i_n)} = {} & [\cos(k\theta) + \cos((k-2\ell)\theta)\cos(2h\theta)][\cos(k\theta) + \cos((r-(2\ell-1))\theta)] \\ & \cdots \cos((k-2i_1)\theta)[\cos(k\theta) - \cos((r-(2i_1-1))\theta)] \\ & [\cos(k\theta) - \cos((k-2(i_1-1))\theta)\cos(2h\theta)][\cos(k\theta) - \cos((r-(2i_1-3))\theta)] \\ & \cdots \cos((k-2i_2)\theta)[\cos(k\theta) + \cos((r-(2i_2-1))\theta)] \\ & [\cos(k\theta) + \cos((k-2(i_2-1))\theta)\cos(2h\theta)][\cos(k\theta) + \cos((r-(2i_2-3))\theta)] \\ & \cdots [\cos(k\theta) + (-1)^n \cos(k-1)\theta)], \end{aligned}$$

*Interchange $\pm$, $a_0/b_0$ to obtain $b_{2\ell}$. Recurrences yield $a_{2\ell+1}$, $b_{2\ell+1}$.*

# Unitarity at distinguished points - $a_i, b_i$ when $k$ even

## Proposition

*If $k$ is even,*

$$a_{2\ell} = \sum_{\substack{(i_1,\dots,i_n)\\ \subseteq(\ell,\dots,1)}} (-1)^n [\sin(2h\theta)]^n \Pi'_{(i_1,\dots,i_n)} \begin{cases} a_0 & \text{if } n \text{ is even,} \\ b_0 & \text{if } n \text{ is odd,} \end{cases} \quad \text{where}$$

$$\begin{aligned}\Pi'_{(i_1,\dots,i_n)} = {} & [\cos(k\theta) + \cos((r-(2\ell-1))\theta)\cos(2h\theta)][\cos(k\theta)+\cos((k-2\ell)\theta)] \\ & \cdots \cos((k-2i_1)\theta)[\cos(k\theta)+\cos((k-2i_1)\theta)] \\ & [\cos(k\theta) - \cos((r-(2i_1-3))\theta)\cos(2h\theta)][\cos(k\theta)-\cos((k-2(i_1-1))\theta)] \\ & \cdots \cos((k-2i_2)\theta)[\cos(k\theta)-\cos((k-2i_2)\theta)] \\ & [\cos(k\theta) + \cos((r-(2i_2-3))\theta)\cos(2h\theta)][\cos(k\theta)+\cos((k-2(i_2-1))\theta)] \\ & \cdots\end{aligned}$$

*Interchange $\pm$, $a_0/b_0$ to obtain $b_{2\ell}$. Recurrences yield $a_{2\ell+1}$, $b_{2\ell+1}$.*

# Eigenvalues of $\rho^h(A(t))$

Claim eigenvalues of $\rho^h(A(t))$ in basis yielding $A_1^h$, $A_2^h$ are:

$$\lambda_h^{\pm}(t) = \prod_{k=-m}^{m} [t \pm \cos(k+h)\theta]$$

$$= (t \pm 1) \prod_{j=1}^{m-h} (t \pm \cos(j\theta))^2 \prod_{j=m-h+1}^{m} (t^2 - \cos^2(j\theta)).$$

True when $h = 0$, since

$$\rho^0(s_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \rho^0(s_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow$$

$$\rho^0(A(t)) = \begin{bmatrix} (t+1)\prod_{k=1}^{m}(t+\cos k\theta)^2 & 0 \\ 0 & (t-1)\prod_{k=1}^{m}(t-\cos k\theta)^2 \end{bmatrix}.$$

# Eigenvalues of $\rho^h(A(t))$

Know $\lambda_h^{\pm}(t)$ are polynomials of degree $2m+1$ in $t$ and divide

$$\det[\rho^h(A(t))] = (t^2-1)\prod_{j=1}^{m}(t^2-\cos^2(j\theta))^2.$$

Find zeroes at distinguished points.
Since $\rho(A(\nu))$ is the restriction of the hermitian form to the $\rho$-isotypic component, $I(\nu)_\rho$ of $I(\nu)$, using $\pi : I(\nu) \twoheadrightarrow L(\nu)$ yields

$$\dim\ker\pi|_{I(\nu)_\rho} = \dim\rho\cdot z(\rho(A(\nu))),$$

where $z(\rho(A(\nu))) = \#$ zero eigenvalues of $\rho(A(\nu))$.
By [K-Ram,2002]

$$2(m-k)+1 = \dim L_{c,k} = 2(2m+1) - \sum_{\rho\in\widehat{W}} \dim\rho\cdot z(\rho(A(\nu_k))).$$

# Eigenvalues of $\rho^h(A(t))$

$$2(m-k)+1 = \dim L_{c,k} = 2(2m+1) - \sum_{\rho \in \widehat{W}} \dim \rho \cdot z(\rho(A(\nu_k)))$$

For all $0 \leq k \leq n$, $z(\rho_1^+(A(\nu_k))) = 0$ and $z(\rho_1^-(A(\nu_k))) = 1$, so

$$m + k = \sum_{h=1}^{m} z(\rho^h(A(\nu_k))), \text{ which implies for } 1 \leq h \leq m,$$

$z(\rho^h(A(\nu_k))) = 1$ or $2$, $\quad z(\rho^h(A(\nu_m))) = 2, \quad z(\rho^h(A(\nu_0))) = 1.$
Since $\ker \pi \oplus L_{c,k} \simeq \mathbb{C}W$,

$$z(\rho^h(A(\nu_k))) = z(\rho^h(A(\nu_{k-1}))) \text{ for all } h \neq k,$$
$$z(\rho^k(A(\nu_k))) = 2 \text{ and } z(\rho^k(A(\nu_{k-1}))) = 1.$$

# Signatures of $\rho^h(A(t))$ at distinguished points

| $t$ | $L\|_{\mathbb{C}W}$ $h=$ | $0$ | $1$ | $2$ | $\cdots$ | $m-2$ | $m-1$ | $m$ |
|---|---|---|---|---|---|---|---|---|
| $\cos m\theta$ | $\rho^+$ | $0,+$ | $0,0$ | $0,0$ | $\cdots$ | $0,0$ | $0,0$ | $0,0$ |
| $\cos(m-1)\theta$ | $\rho^+\oplus\rho^1$ | $0,+$ | $0,+$ | $0,0$ | $\cdots$ | $0,0$ | $0,0$ | $0,0$ |
| $\cos(m-2)\theta$ | $\rho^+\oplus\bigoplus_{j=1}^{2}\rho^j$ | $0,+$ | $0,+$ | $0,+$ | $\cdots$ | $0,0$ | $0,0$ | $0,0$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\cos 2\theta$ | $\rho^+\oplus\bigoplus_{j=1}^{m-2}\rho^j$ | $0,+$ | $0,+$ | $0,+$ | $\cdots$ | $0,+$ | $0,0$ | $0,0$ |
| $\cos\theta$ | $\rho^+\oplus\bigoplus_{j=1}^{m-1}\rho^j$ | $0,+$ | $0,+$ | $0,+$ | $\cdots$ | $0,+$ | $0,+$ | $0,0$ |
| $1$ | $\rho^+\oplus\bigoplus_{j=1}^{m}\rho^j$ | $0,+$ | $0,+$ | $0,+$ | $\cdots$ | $0,+$ | $0,+$ | $0,+$ |

# Eigenvalues of $\rho^h(A(t))$

Pattern of signatures yields:

$$\prod_{j=m-h+1}^{m} (t - \cos(j\theta)) \text{ divides } \lambda_h^+(t),$$

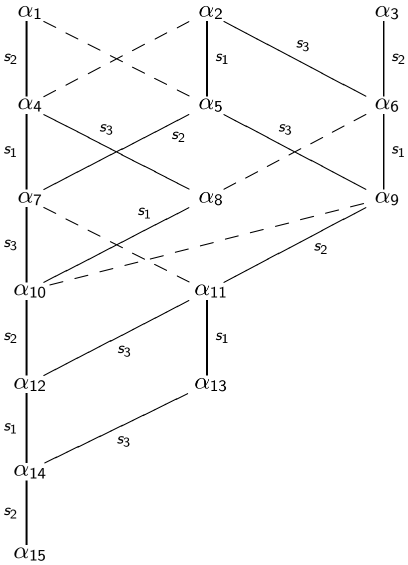
$$(t-1)\prod_{j=1}^{m-h}(t-\cos(j\theta))^2 \prod_{j=m-h+1}^{m} (t - \cos(j\theta)) \text{ divides } \lambda_h^-(t)$$

Repeating for $t = -\cos(k\theta)$ and using the Iwahori-Matsumoto involution ($IM(t_w) = (-1)^{\ell(w)} t_w, IM(\omega) = -\omega$) which preserves unitarity yields

$$\lambda_h^\pm(t) = \prod_{k=-m}^{m} [t \pm \cos(k+h)\theta]$$

$$= (t \pm 1)\prod_{j=1}^{m-h}(t \pm \cos(j\theta))^2 \prod_{j=m-h+1}^{m} (t^2 - \cos^2(j\theta)).$$

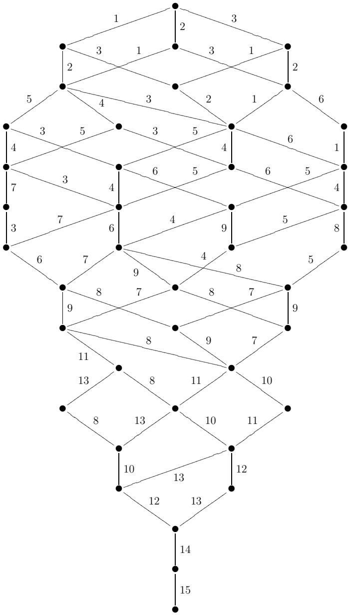
# Positive roots, actions of $s_i$, and the root order for $H_3$



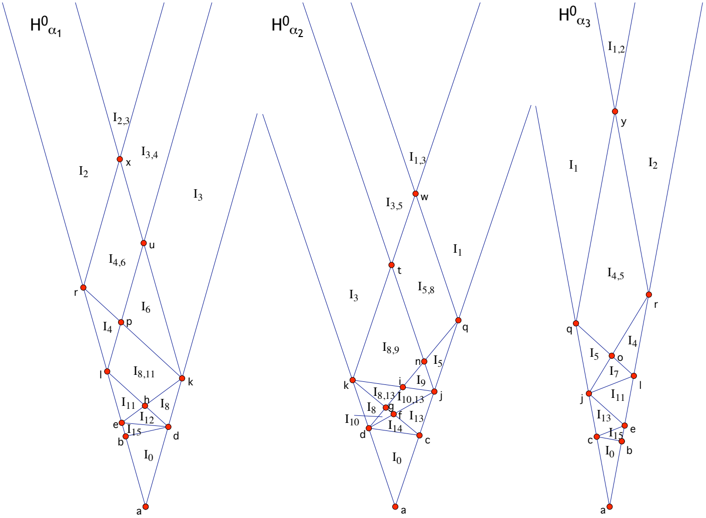

where for $\tau = \frac{1+\sqrt{5}}{2}$,

$$
\begin{aligned}
\alpha_4 &= \alpha_1 + \tau\alpha_2 \\
\alpha_5 &= \tau\alpha_1 + \alpha_2 \\
\alpha_6 &= \alpha_2 + \tau\alpha_3 \\
\alpha_7 &= \tau\alpha_1 + \tau\alpha_2 \\
\alpha_8 &= \alpha_1 + \tau\alpha_2 + \tau\alpha_3 \\
\alpha_9 &= \tau\alpha_1 + \alpha_2 + \alpha_3 \\
\alpha_{10} &= \tau\alpha_1 + \tau\alpha_2 + \tau\alpha_3 \\
\alpha_{11} &= \tau\alpha_1 + (\tau+1)\alpha_2 + \alpha_3 \\
\alpha_{12} &= \tau\alpha_1 + (\tau+1)\alpha_2 + \tau\alpha_3 \\
\alpha_{13} &= (\tau+1)\alpha_1 + (\tau+1)\alpha_2 + \alpha_3 \\
\alpha_{14} &= (\tau+1)\alpha_1 + (\tau+1)\alpha_2 + \tau\alpha_3 \\
\alpha_{15} &= (\tau+1)\alpha_1 + 2\tau\alpha_2 + \tau\alpha_3
\end{aligned}
$$

# Ideals in the positive root poset or 3-dim cells for $H_3$



# Reflecting walls of $\overline{C}$ for $H_3$



# Intertwining operator for $H_3$

For $W = H_3$, the choice $w_0 = (s_1 s_2 s_1 s_2 s_3)^3$ induces the ordering

$$\{\alpha_1, \alpha_5, \alpha_7, \alpha_4, \alpha_{13}, \alpha_2, \alpha_{11}, \alpha_{15}, \alpha_{14}, \alpha_{12}, \alpha_9, \alpha_{10}, \alpha_8, \alpha_6, \alpha_3\}.$$

For each $\nu \in V_0^+ = C$,

$$A(\nu) = \left(1 + \langle \nu, \alpha_1^\vee \rangle s_1\right) \left(1 + \langle \nu, \alpha_5^\vee \rangle s_2\right) \cdots \left(1 + \langle \nu, \alpha_6^\vee \rangle s_2\right) \left(1 + \langle \nu, \alpha_3^\vee \rangle s_3\right)$$

Use $W$-graphs to obtain irreducible matrix representations $\rho_1^\pm, \rho_3^\pm, \bar{\rho}_3^\pm, \rho_4^\pm, \rho_5^\pm$ that are conjugate to unitary matrices.

# Techniques - computational in Maple

1. Understand geometry of cells in $\overline{C}$
   a. Solve all systems and select dominant vertices.
      Used Cramer's rule and symbolic simplification.
   b. Used product of incidence-type matrices to match vertices to 3-dim cells.
   c. Used incidence matrix for vertices of each 3-dim cell to detect 1-dim and 2-dim boundary cells.
2. Analyzed signs of eigenvalues for each $\rho(A(\nu))$ where $\nu$ is a vertex or average of vertices (use continuity of $\rho(A(\nu))$)
   a. Found $\rho(A(\nu))$ and characteristic polynomial $\chi_{\rho,\nu}$ over $\mathbb{Q}[\sqrt{5}]$
   b. Analyzed signs of coefficients in $\chi_{\rho,\nu}$ by exact arithmetic.
   c. Descartes' Rule of Signs detected/ruled out eigenvalues $< 0$.

# The support of the generic unitary dual for $\mathbb{H}(H_3)$

- Three 3-dim cells ($\nu = \nu_1\omega_1^\vee + \nu_2\omega_2^\vee + \nu_3\omega_3^\vee$)

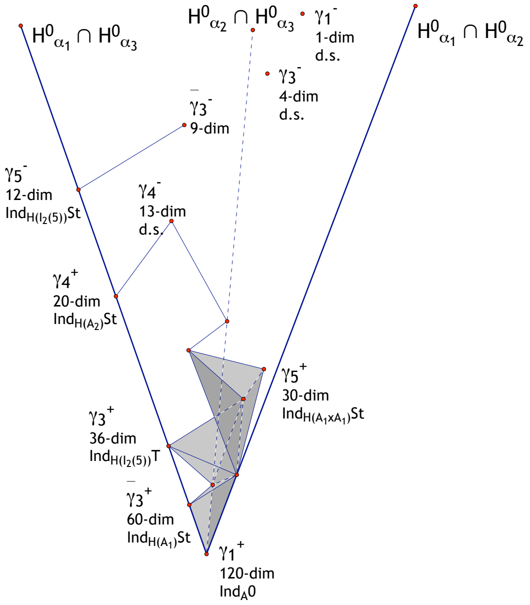
$$\begin{aligned}
I_\emptyset \quad \text{is} \quad & 0 \leq (\tau+1)\nu_1 + 2\tau\nu_2 + \tau\nu_3 \leq 1 \\
I_{14} \quad \text{is} \quad & 1 \leq (\tau+1)\nu_1 + (\tau+1)\nu_2 + \tau\nu_3, \\
& 0 \leq (\tau+1)\nu_1 + (\tau+1)\nu_2 + \nu_3 \leq 1, \\
& 0 \leq \tau\nu_1 + (\tau+1)\nu_2 + \tau\nu_3 \leq 1 \\
I_{10} \quad \text{is} \quad & 1 \leq \tau\nu_1 + \tau\nu_2 + \tau\nu_3, \\
& 0 \leq \nu_1 + \tau\nu_2 + \tau\nu_3 \leq 1, \\
& 0 \leq (\tau+1)\nu_1 + (\tau+1)\nu_2 + \nu_3 \leq 1
\end{aligned}$$

- Four 1-dim cells, $0 \leq a \leq 1$.

$$\begin{aligned}
&\nu_1 = (1-a)(\tau-1), && \nu_2 = a(2\tau-3), && \nu_3 = a(2-\tau) \\
&\nu_1 = a(3-2\tau)+\tau-1, && \nu_2 = a(2\tau-3), && \nu_3 = a(-3\tau+5) \\
&\nu_1 = a(2-\tau), && \nu_2 = a(\tau-2)+\tau-1, && \nu_3 = a(-3\tau+5) \\
&\nu_1 = (2-\tau)(1-a), && \nu_2 = (2-\tau)(1+\tau a), && \nu_3 = (2-\tau)(1-a)
\end{aligned}$$

- Two isolated vertices: $\nu_3^- = (2-\tau, 2-\tau, 1)$ and $\nu_1^- = (1,1,1)$.

# The support of the generic unitary dual for $\mathbb{H}(H_3)$



# $I_2(2m)$

Two distinct parameters, $c_1$, $c_2$ - one for each orbit of roots and $n^2+n+3$ distinct cases for $c_1/c_2$.
Since $w_0 = (s_1 s_2)^m$ is rotation by $\pi$,

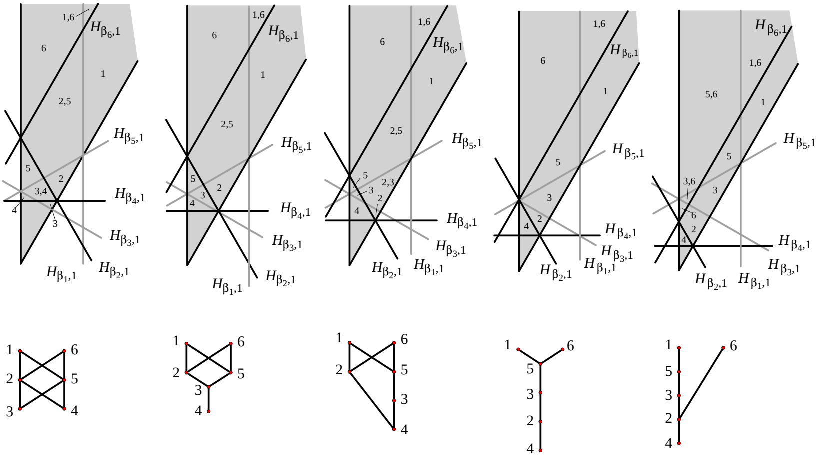
$$V_0^+ = \{\nu \in \mathfrak{h}_{\mathbb{R}}^* \mid w_0\nu = -\nu\} \cap \overline{C} = \overline{C}.$$

$\{$2-dim regions in $\mathfrak{h}_{\mathbb{R}}^* \setminus \left(\cup_{i=1}^n H_{\alpha_i}^0, H_{\alpha_i}^{c_i}\right)\} \leftrightarrow \{$ideals in $(R, \leq)\}$
Build poset $(\mathcal{I}, \subseteq)$ of ideals. Label edges $I \xrightarrow{c_\alpha} I \cup \{\alpha\}$.

Conjecture

*Support of the unitary 0-complementary series for $\mathbb{H}(I_2(2m))$ is $\overline{\cup_I R_I}$ where $I$ can be reached in the poset of ideals by a path of even length with same labeled edges, starting from $\emptyset$, i.e., starting in $R_\emptyset$, cross evenly many $H_{\alpha_i}^{c_i}$ in same orbit as move along ray in $C$.*

# Dominant regions and positive root order posets for $I_2(6)$




$\frac{\sqrt{3}}{2} < r < \frac{2}{\sqrt{3}}$ $(r = 1)$

$r = \frac{2}{\sqrt{3}}$

$\frac{2}{\sqrt{3}} < r < \sqrt{3}$

$r = \sqrt{3}$

$\sqrt{3} < r$