Hermitian and Unitary Representations for Affine Hecke Algebras

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Introduction

This talk is about aspects of representation theory that David Vogan (I think) will explain in his talks in the context of p-adic groups.

As the title indicates, the context is that of the affine graded Hecke algebra. Some of the results are well known in the group case. One of the purposes is to present a self contained theory in for graded affine algebras.

Most of the results are inspired by those of real groups (Lefschetz principle), and [BM3].

Work still in progress

The Unitarity Problem, P-adic Groups

NOTATION

- G is the rational points of a linear connected reductive group over a local field $\mathbb{F}_p \supset \mathcal{R}_p \supset \mathcal{P}_p$.
- The Hecke algebra is

  $$\mathcal{H}(G) := \{f : G \longrightarrow \mathbb{C}, f \text{ compactly supported, locally constant } \}$$

- A representation (π, U) is called hermitian if U admits a hermitian invariant form, and unitary, if U admits a $G-$invariant positive definite inner product.
- It is called admissible if $Stab_G(v)$ for any vector $v \in U$ is open, and $U^{\mathcal{K}}$ is finite dimensional for any compact open subgroup $\mathcal{K} \subset G$.

PROBLEM

Classify all irreducible unitary representations of G.

It is enough (Harish-Chandra) to solve an

ALGEBRAIC PROBLEM:

Classify the unitary dual for irreducible admissible $\mathcal{H}(G)-$modules.

$\mathcal{H}(G)$ is an algebra under convolution, and is endowed with a conjugate linear involutive anti-automorphism

$$(f)^\star(x) := \overline{f(x^{-1})}.$$

Hermitian: $\langle \pi(f)v_1, v_2\rangle = \langle v_1, \pi(f^\star)v_2\rangle$.

Unitary: Hermitian plus $\langle\ ,\ \rangle >> 0$ (*i.e.* positive definite).

A Reduction

According to results of Bernstein, the category of admissible representations breaks up into blocks. Further results, starting with Borel-Casselmann, Howe-Moy, Bushnell-Kutzko and many others (J. Kim ...), imply that each block is equivalent to a category of finite dimensional representations of an Iwahori-Hecke type algebra.

Main Example: (the prototype of the results above, due to Borel-Casselman).

- G split, $B = AN \subset G$ a Borel subgroup.
- $\mathcal{I} \subset G$ an Iwahori subgroup
- $\mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$ the Iwahori-Hecke algebra of $\mathcal{I}-$biinvariant functions in $\mathcal{H}(G)$.

Theorem (Borel-Casselmann). *The category of admissible representations all of whose subquotients are generated by their $\mathcal{I}-$invariant vectors is equivalent to the category of finite dimensional $\mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})-$modules via the functor*

$$U \mapsto U^{\mathcal{I}}.$$

It is clear that the functor takes a unitary module to a unitary module.

Theorem (B-Moy). *A module (π, U) is unitary if and only if $(\pi^{\mathcal{I}}, U^{\mathcal{I}})$ is unitary.*

The algebra $\mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$ can be described by generators and relations. In the case of a more general block, the analogous algebra appearing in the B-C result is more complicated. Most (if not all) cases are covered by a generalization of the B-Moy theorem by B-Ciubotaru.

The Graded Affine Hecke Algebra

- $\Phi = (V, R, V^\vee, R^\vee)$ an $\mathbb{R}$−root system, reduced.
- W Weyl group.
- $\Pi \subset R$ simple roots, R^+ positive roots.
- $k : \Pi \to \mathbb{R}$ a function such that $k_\alpha = k_{\alpha'}$ whenever $\alpha, \alpha' \in \Pi$ are W-conjugate.

Definition (Graded Affine Hecke Algebra)**.**
$\mathbb{H} = \mathbb{H}(\Phi, k) \cong \mathbb{C}[W] \otimes S(V_\mathbb{C})$ *such that*

(i) $\mathbb{C}[W]$ *and* $S(V_\mathbb{C})$ *have the usual algebra structure,*

(ii) $\omega t_{s_\alpha} = t_{s_\alpha} s_\alpha(\omega) + k_\alpha \langle \omega, \check{\alpha} \rangle$ *for all* $\alpha \in \Pi$, $\omega \in V_\mathbb{C}$.

Star Operations

A star operation is a conjugate linear involutive algebra anti-automorphism κ. Without one, if (π, U) is a module, so cannot define an action on the hermitian dual U^h, and we cannot talk about hermitian modules, let alone unitary ones.

(π, U) gives rise to (π^κ, U^h) by the formula

$$\left(\pi^\kappa(h)f\right)(v) := f(\pi(\kappa(h))v)$$

(π, U) admits a $\kappa-$invariant sesquilinear form if and only if there is a ($\mathbb{C}-$linear) equivariant map $\iota : (\pi, U) \longrightarrow (\pi^\kappa, U^h)$. The associated form is

$$\langle h_1, h_2 \rangle := \iota(h_1)(h_2).$$

The form is *hermitian* if $\iota^h : U \subset \left(U^h\right)^h \longrightarrow U^h$ coincides with ι.

$\mathbb{H}$ has a natural κ which we will denote by $\bullet$:

$$(t_w)^\bullet = t_{w^{-1}}, \qquad (\omega)^\bullet := \overline{\omega},\ \omega \in V_{\mathbb{C}}.$$

(Recall that $V_{\mathbb{C}}$ is the complexification of the real vector space V). However if $\mathbb{H}$ is obtained from a $p-$adic group, there is an *induced* star, and it is not $\bullet$.

It is not far off though; it should satisfy:

(i) $\kappa(t_w) = t_{w^{-1}}$,

(ii) $\kappa(\omega) \subset \mathbb{C}[W] \cdot V_{\mathbb{C}}$ for $\omega \in V_{\mathbb{C}}$.

Theorem (B-Ciubotaru)**.** *Assume the root system Φ is simple. The only involutive anti automorphisms κ satisfying (i) and (ii) are $\bullet$ and $\star$ determined by*

$$\omega^\star = t_{w_0}\overline{(-w_0\omega)}t_{w_0},$$

where $w_0 \in W$ is the long Weyl group element.

We define $\mathcal{A} : \mathbb{H} \longrightarrow \mathbb{H}$ to be the automorphism given by

$$\mathcal{A}(t_w) := t_{w_0ww_0}, \qquad \mathcal{A}(\omega) := -w_0(\omega).$$

$\mathcal{A}(h^\bullet) = (\mathcal{A}(h))^\bullet$ and $\star = \mathrm{Ad}\, t_{w_0} \circ \mathcal{A} \circ \bullet$.

In all examples we know, $\star$ is the star operation coming from the group.

Sketch of Proof

κ is involutive automorphism, $\kappa(t_w) = t_w$ and (ii) from before.

1. $\kappa(t_w) = t_w, \quad w \in W; \ \kappa(\omega) = c_0\omega + \sum_{y\in W} g_y(\omega)t_y, \quad \omega \in V_{\mathbb{C}}$, where $g_y : V_{\mathbb{C}} \to \mathbb{C}$, $y \in W$, are linear.
2. $t_{s_\alpha}\omega - s_\alpha(\omega)t_{s_\alpha} = k_\alpha(\omega, \alpha^\vee)$ implies

   $g_{s_\alpha y s_\alpha}(\omega) = g_y(s_\alpha(\omega)), \ \text{for all } \alpha \in \Pi, y \in W, y \neq s_\alpha, \ \text{and } \omega \in V_{\mathbb{C}}$

   $g_{s_\alpha}(\omega) - g_{s_\alpha}(s_\alpha(\omega)) = k_\alpha(1 - c_0)(\omega, \alpha^\vee).$
3. $\kappa^2 = id$ implies $c_0^2 = 1$.
4. If $c_0 = 1$, we conclude $g_y = 0$, so $\kappa = \bullet$.
5. If $c_0 = -1$ and $-w_0 = id$, $\mathrm{Ad}\, t_{w_0} \circ \kappa$ is another κ, but has $c_0 = 1$. So $\kappa = \star$.

Otherwise need another page.

Langlands Classification

Definition. *A module* (σ, U) *is called tempered (modulo the center) if all the weights* η *of* $V_{\mathbb{C}}$ *satisfy* $Re\langle\varpi_\alpha, \eta\rangle \leq 0$ *for all* $\alpha \in \Pi$, ϖ_α *the corresponding fundamental weight.*

Let $\Pi_M \subset \Pi$ be a subset of the simple roots.

- $\mathbb{H}_M := span\{t_\alpha,\ \omega\}$, $\alpha \in \Pi_M$, $\omega \in V_{\mathbb{C}}$.
- $V_M \subset V$ the kernel of the $\check{\alpha}$ with $\alpha \in \Pi_M$.
- $X(M, \sigma_0, \nu) := \mathbb{H} \otimes_{\mathbb{H}_M} [U_{\sigma_0} \otimes \mathbb{C}_\nu]$
  the standard module attached to a tempered module σ_0 of (the semisimple part of) $\mathbb{H}_M$ and a character ν of V_M.

Theorem (Langlands Classification)**.**

(i) *If* $Re\langle\nu,\alpha\rangle > 0$ *for all* $\alpha \in \Pi\backslash\Pi_M$, *then* $X(M,\sigma_0,\nu)$ *has a unique irreducible quotient* $L(M,\sigma_0,\nu)$.

(ii) *Every irreducible module is isomorphic to an* $L(M,\sigma_0,\nu)$.

(iii) $L(M,\sigma_0,\nu) \cong L(M',\sigma_0',\nu')$ *if and only if the data are conjugate* $(M = M',\ \sigma_0 \cong \sigma_0',\ \nu = \nu')$.

(iv) *Denote by* aM *the Levi component corresponding to* $\mathcal{A}(\Pi_M)$ *and* $a\sigma_0,\ a\nu$ *the transfers of* σ_0, ν *to* $\mathbb{H}_{aM}$ *(*$a\nu$ *is conjugate by* w_0*, negative). Then* $L(M,\sigma_0,\nu)$ *is the image of an intertwining operator*

$$\begin{aligned} A_{w^0} :& X(M,\sigma_0,\nu) \longrightarrow X(aM,a\sigma_0,a\nu),\\ & h\otimes v \mapsto hR_{w^0}\otimes a(v) \end{aligned}$$

w^0 *is the minimal element in the coset* $w_0W(aM)$, *and* $R_{w^0} \in \mathbb{H}$ *is explicit defined as follows.*

- $\alpha \in \Pi$, $r_\alpha := t_{s_\alpha}\alpha - k_\alpha$,
- $w = s_1 \cdot \dots \cdot s_k$, $R_w := \prod r_{\alpha_i}$.

R_w does not depend on the particular minimal decomposition of w into simple reflections. Its main property is that

$$R_w \omega = w(\omega) R_w.$$

Induced Hermitian Modules

- (σ, U) a representation of $\mathbb{H}_M$, $X(M,\sigma) := \mathbb{H} \otimes_{\mathbb{H}_M} \sigma$ with action $\pi(h)h_1 \otimes v := hh_1 \otimes v$.
- $(\sigma^{\bullet_M}, U^h)$ and $(\sigma^{\star_M}, U^h)$ the representations on the hermitian dual space U^h.
- $(\pi^{\bullet}, X(M,\sigma)^h)$ and $(\pi^{\star}, X(M,\sigma)^h)$ the representations on the hermitian dual.

Note that $X(M,\sigma)^h \cong \mathrm{Hom}_{\mathbb{H}_M}\left[\mathbb{H}, \sigma^h\right]$ in a canonical way

$$\Psi \in \mathrm{Hom}_{\mathbb{H}_M}\left[\mathbb{H}, \sigma^h\right] \mapsto \widetilde{\Psi}[h \otimes v] = \Psi(h)(v)$$

(Here $\mathbb{H}$ is viewed as a right $\mathbb{H}_M$ module).

Since $\dim X(M,\sigma) < \infty$, $X(M,\sigma)^h \cong \mathbb{H} \otimes_{\mathbb{H}_M} \sigma^h$ (with the action $\bullet_M$ or $\star_M$), has a basis $\{t_x^h \otimes v_i^h\}$ where t_x^h is a dual basis to $\{t_x\}_{x \in W/W(M)}$ and $\{v_i^h\}$ dual to a basis $\{v_i\}$ of U.

Theorem. *[B-Ciubotaru] The map*

$$\Phi(t_x^h \otimes v^h) := t_{xw^0_{aM}} \otimes av^h$$

is an $\mathbb{H}-$equivariant isomorphism between $\left(\pi_\sigma^\bullet, X(M,\sigma)^h\right)$ and $\left(\pi_\sigma, X(aM, a\sigma^h)\right)$ where the action on σ^h is given by $\bullet_{a(M)}$.

Similarly for $\star$.

Example. $\Pi_M = \emptyset$. *The standard module os $X(\nu)$ the full principal series.*

$$(\pi^\bullet, X(\nu)^h) \cong (\pi, X(w_0\overline{\nu}))$$
$$(\pi^\star, X(\nu)^h) \cong (\pi, X(-\overline{\nu}))$$

Sesquilinear Forms

A $\bullet-$invariant sesquilinear form on $X(M,\sigma)$ is equivalent to defining an $\mathbb{H}$-equivariant map

$$\mathcal{I} : (\pi, X(M,\sigma)) \longrightarrow (\pi^{\bullet}, X(M,\sigma)^h). \tag{1}$$

We call $\mathcal{I}$ hermitian if $\mathcal{I}^h = \mathcal{I}$ or equivalently $\mathcal{I}(v)(w) = \overline{\mathcal{I}(w)(v)}$, for all $v, w \in X(M,\sigma)$. Recall M, aM, and
$w_0 = w^0 w_{0,aM)} = w_{0,M} w^0$ with w^0 minimal in $w_0 W_{aM}$.

To simplify notation, write $\widetilde{M} = aM$, and $R^0 := R_{w^0}$. Assume that there is an $\mathbb{H}_M$-equivariant isomorphim $\iota : (\sigma, U_\sigma) \longrightarrow (\sigma^{\bullet}, U_\sigma^h)$ defining an invariant hermitian form $\langle\ ,\ \rangle$ on (σ, U_σ). The same map gives an isomorphism $\iota_a : (a\sigma, U_\sigma) \longrightarrow (a\sigma^{\bullet}, U_\sigma^h)$.

Write $R^0 = \sum t_z m^0_{\widetilde{z}}$ with $\widetilde{z}$ minimal in $\widetilde{z}W(\widetilde{M})$ and $m^0_{\widetilde{z}} \in \mathbb{H}_{\widetilde{M}}$.

Proposition. *Suppose* (σ, U_σ) *has* $\bullet$*-invariant hermitian form* $\langle\ ,\ \rangle_\sigma$. *The form*

$$\langle h_1 \otimes v_1, h_2 \otimes v_2 \rangle_\bullet = \langle a\sigma[\epsilon_{\widetilde{M}}(t_{(w^0)^{-1}} h_2^\bullet h_1 R^0)] v_1, v_2 \rangle_\sigma.$$

on $X(M, \sigma)$ *is* $\bullet$*-invariant and sesquiliniar.*

Similar result for $\star$.

Assume ν is real, *i.e.* $\nu \in V_M^\vee$.

Corollary. $L(M, \sigma_0, \nu)$ *admits a nondegenerate hermitian form for*

$\bullet$**:** *any* $\nu \in V_M^\vee$,

$\star$**:** *if and only if there exists* $w \in W$ *such that* $w\nu = -\nu$, *and* $w \circ \sigma_0 \cong \sigma_0$ *(in this case* $aM = M$*).*

Spherical Principal Series

- $\nu \in V_{\mathbb{C}}^{\vee}$, $A := S(V_{\mathbb{C}})$
- $X(\nu) := \mathbb{H} \otimes_A \mathbb{C}_\nu$.
- $\langle h_1, h_2 \rangle_{\bullet,\nu} = \epsilon_A\,(t_{w_0} h_2^{\bullet} h_1 R_{w_0})\,(w_0\nu)$.
- $\langle h_1, h_2 \rangle_{\star,\nu} = \epsilon_A\,(h_2^{\star} h_1 R_{w_0})\,(w_0\nu)$.

One Consequence

The classification results of Kazhdan-Lusztig imply that standard modules have lowest $W-$types. In the case of $Re\langle\nu,\alpha\rangle > 0$ they determine the Langlands quotient $L(M,\sigma_0,\nu)$ (the Langlands quotient is the unique irreducible subquotient containing *all* the lowest $W-$types with full multiplicity occuring in $X(M,\sigma_0,\nu)$).

Proposition (B-Ciubotaru)**.** *When $L(M,\sigma_0,\nu)$ is hermitian, the nondegenerate form can be normalized so that:*

- *the $\bullet$-form is positive on the lowest $W-$types,*
- *Let $\deg(\mu)$ be the lowest degree so that μ occurs in the harmonics of $S(V_{\mathbb{C}})$. When t_{w_0} is central, the $\star-$form on a lowest $K-$type μ is given by $(-1)^{\deg(\mu)}$. The general formula is a little more complicated.*

to be continued ...

Thank you for the opportunity to present these results, and the work you put into organizing this conference!

References

[BM1] D. Barbasch, A. Moy, *A unitarity criterion for p-adic groups*, Invent. Math. 98, 1989, 19–38.

[BM2] ______, *Reduction to real infinitesimal character in affine Hecke algebras*, J. Amer. Math. Soc. 6(3), 1993, 611–635.

[BM3] ______, *Unitary spherical spectrum for classical p-adic groups* Actae Applicandae Mathematicae, vol. 44, no 1-2, 1996, 3–37

[Ev] S. Evens, *The Langlands classification for graded Hecke algebras*, Proc. Amer. Math. Soc. 124 (1996), no. 4, 1285–1290.

[KL] D. Kazhdan, G. Lusztig *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. 87, 1987, 153–215.

[Lu1] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. 2, 1989, 599–635.