Spherical nilpotent orbits and representations of semi-simple Lie groups

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Notation

- $\mathfrak{g}$ real simple Lie algebra, Cartan involution θ.
- Cartan decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
- Complexifications: $\mathfrak{g}_{\mathsf{c}}$, $\mathfrak{k}_{\mathsf{c}}$ and $\mathfrak{p}_{\mathsf{c}}$
- G_{c} is a connected linear group with Lie algebra $\mathfrak{g}_{\mathsf{c}}$.
- G, K, and K_{c} conn. subgrps corr. to $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{k}_{\mathsf{c}}$.
- Group actions: G_{c} on $\mathfrak{g}_{\mathsf{c}}$, G on $\mathfrak{g}$, K_{c} on $\mathfrak{p}_{\mathsf{c}}$

Basic definitions

Definition

- *$ad(x)(z) := [x,\ z],\ \ x,\ z \in \mathfrak{g}_{\mathbf{c}}$.*
- *x is nilpotent (resp., semi-simple) if $ad(x) : \mathfrak{g}_{\mathbf{c}} \to \mathfrak{g}_{\mathbf{c}}$ is nilpotent (resp., semi-simple)*
- *If x is semi-simple, $\mathfrak{g}_{\mathbf{c}}(x; j) =$ the j-eigenspace of $ad(x)$.*
- *$\mathfrak{p}_{\mathbf{c}}(x; j)$ and $\mathfrak{k}_{\mathbf{c}}(x; j)$ are defined similarly for $x \in \mathfrak{k}_{\mathbf{c}}$ and semi-simple.*

Definition

The non-zero elements $A,\ B,\ C$ in a real Lie algebra are said to form an $sl(2)$-triple if $[B,\ C] = A$, $[A,\ B] = 2B$, and $[A,\ C] = -2C$. A is called the neutral element of the triple.

Some $sl(2)$-triples

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

and

$$\left\{ \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \right\}$$

Nilpotent orbits and their classification

Consider the following sets of nilpotents: $N(\mathfrak{g}_{\mathsf{c}})$, $N(\mathfrak{g})$, and $N(\mathfrak{p}_{\mathsf{c}})$.

and corresponding sets of orbits: $N(\mathfrak{g}_{\mathsf{c}})/G_{\mathsf{c}}$, $N(\mathfrak{g})/G$, and $N(\mathfrak{p}_{\mathsf{c}})/K_{\mathsf{c}}$.

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1. $N(\mathfrak{g}_{\mathsf{c}})/G_{\mathsf{c}} \leftrightarrow G_{\mathsf{c}}$*-conjugacy classes of* $sl(2, \mathbb{C})$ *subalgebras of* $\mathfrak{g}_{\mathsf{c}}$
2. $N(\mathfrak{g})/G \leftrightarrow G$*-conjugacy classes of* $sl(2)$*-triples* $\{H, E, F\}$ *in* $\mathfrak{g}$
3. $N(\mathfrak{p}_{\mathsf{c}})/K_{\mathsf{c}} \leftrightarrow$
   K_{c}*-conjugacy classes of* $sl(2)$*-triples* $\{x, e, f\}$ *(with* $e, f \in \mathfrak{p}_{\mathsf{c}}$*)*

Co-ordinate rings of nilpotent orbits in $N(\mathfrak{p}_{\mathsf{c}})$

- $\mathcal{O} = K_{\mathsf{c}} \cdot e$ is a quasi-affine variety in $\mathfrak{p}_{\mathsf{c}}$.
- $\overline{\mathcal{O}}$:= the Zariski closure of $\mathcal{O}$ in $\mathfrak{p}_{\mathsf{c}}$.
- If $f \in R[\mathcal{O}]$, $k \in K_{\mathsf{c}}$ the group action is $(k \cdot f)(z) = f(k^{-1} \cdot z)$, for $z \in \mathcal{O}$.
- $R[\mathcal{O}]$, $R[\overline{\mathcal{O}}]$, and $R[\overline{\mathcal{O}}^n]$ are completely reducible K_{c}-modules, e.g.,

$$R[\mathcal{O}] = \sum_{\mu \in \widehat{K}} m_{\mathcal{O}}(\mu) V_{\mu}, \ \textit{where } m_{\mathcal{O}}(\mu) < \infty$$

Co-ordinate rings of nilpotent orbits in $N(\mathfrak{p}_{\mathbb{C}})$

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Definition

$\mathcal{O}$ is spherical if $m_{\mathcal{O}}(\mu) = 0$ or 1 $\forall \mu \in \widehat{K}$.

Example

$\mathfrak{g} = sl(3,\ \mathbb{R})$, $K_{\mathrm{c}} = SO(3,\ \mathbb{C})$

$$\mathcal{O} = \text{minimal orbit} \leftrightarrow 2+1 \leftrightarrow \begin{bmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R[\mathcal{O}] = R[\overline{\mathcal{O}}] = R[\overline{\mathcal{O}}^n] \simeq \sum_{j=0}^{\infty} H_{2j}.$$

$\forall$ non-negative integer n, $H_n =$ unique irr. repn. of $SO(3,\ \mathbf{C})$ s.t. $\dim H_n = 2n+1$.

$H_n = \mathbb{C}$-span of the real valued harmonic polynomials of degree n in 3 variables.

Spherical nilpotent orbits

Proposition

$\mathcal{O}$ is spherical for K_{c} $\iff$ a Borel subgroup of K_{c} has a dense orbit in $\mathcal{O}$.

It follows that the varieties $\mathcal{O}$, $\overline{\mathcal{O}}$, and $\overline{\mathcal{O}}^n$ are each spherical or none are spherical.

([McGovern, 1994] and [Panyushev, 1994]

Classification of spherical nilpotent G-orbits for $\mathfrak{g}$ simple and complex.

Panyushev's criterion:

$$\mathcal{O} = G \cdot e \text{ is spherical} \iff \text{height}(e) = 2 \text{ or } 3$$

where "height(e)" = largest eigenvalue of a neutral element for $\mathcal{O}$

Spherical nilpotent orbits in $N(\mathfrak{p}_{\mathsf{c}})/K_{\mathsf{c}}$

$\mathcal{O}_{\mathbb{C}}$ spherical for G_{c} $\Rightarrow$ each K_{c}-orbit in $\mathcal{O}_{\mathbb{C}} \cap \mathfrak{p}_{\mathsf{c}}$ is spherical for K_{c}.

Spherical nilpotent orbits in $N(\mathfrak{p}_c)/K_c$

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[K, 2004]

Classification of spherical nilpotent K_c-orbits in $\mathfrak{p}_c$, when $\mathfrak{g}$ simple and real.

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- What can spherical nilpotent orbits tell us about Harish Chandra modules?

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Classification of spherical nilpotent K_c-orbits in $\mathfrak{p}_c$, when $\mathfrak{g}$ simple and real.

- What can spherical nilpotent orbits tell us about Harish Chandra modules?
- A lot of information is contained in $R[\overline{\mathcal{O}}]$

From now on assume that $\mathcal{O}$ has height two.

Facts about $R[\overline{\mathcal{O}}]$ for small spherical orbits

Proposition

Fix $\mathfrak{t} \subset \mathfrak{k}$, $\mathfrak{t}_{\mathbf{c}} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$, *a Borel subalgebra* $\mathfrak{b}_{\mathbf{c}} = \mathfrak{t}_{\mathbf{c}} \oplus \mathfrak{n}_{\mathbf{c}} \subset \mathfrak{k}_{\mathbf{c}}$.

1. $\overline{\mathcal{O}}$ *is normal.*
2. $R[\overline{\mathcal{O}}]^{\mathfrak{n}_{\mathbf{c}}} = \mathbb{C}[u_1, \ldots, u_r]$; *each* u_i *is homogeneous of weight* γ_i.
3. $\exists$ *a subspace* $W \subset \mathfrak{p}_{\mathbf{c}}$ *that is* $K_{\mathbf{c}}$*-conjugate to either* $\mathfrak{p}_{\mathbf{c}}(x; 2)$ *or* $\mathfrak{p}_{\mathbf{c}}(x; -2)$ *such that* $\forall\ i,\ u_i \in S(W)$.

Discussion

- The proof uses ideas of Hesselink and Panyushev applied to the standard desingularisation of $\overline{\mathcal{O}}$.
- Same result holds in a slightly more general context (e.g., the principal nilpotent orbit of $su(2,\ 1)$).
- Similar and related results have been proven for many of these orbits by many other authors including Binegar, and jointly by the Nishiyama, Ochiai, and Zhu.

An application to Harish Chandra modules

Keep the notation from the preceding proposition and in addition:

$$\Gamma(\overline{\mathcal{O}}) := \text{the monoid in } i\mathfrak{t}^* \text{ of } K\text{-types in } R[\overline{\mathcal{O}}]$$

$$\mathcal{C}(\overline{\mathcal{O}}) = \mathbb{R}^+\Gamma(\overline{\mathcal{O}}) := \text{the span of } \Gamma(\overline{\mathcal{O}}) \text{ over } \mathbb{R}^+$$

Theorem

Assume that X is an irreducible $(\mathfrak{g}_{\mathbf{c}},\ K)$-module and the associated variety of X equals $\overline{\mathcal{O}}$. Then, the asymptotic directions of the K-types of X are the same as those of the cone $\mathcal{C}(\overline{\mathcal{O}})$.

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Knapp, Trapa and many other authors have noticed this behavior.

Definition

Let $\sigma_1 = \sigma(u_1), \ \ldots, \ \sigma_r = \sigma(u_r)$ be the images of the u_i in $U(\mathfrak{g}_{\mathbf{c}})$ under symmetrization.

Key idea of proof: $\sigma_1, \ \ldots, \ \sigma_r$ act injectively on X because W satisfies a certain technical condition studied by Gyoja and Yamashita.

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Example

$\mathfrak{g} = sl(4,\ \mathbf{R})\quad \mathfrak{k} = so(4) = so(3) \oplus so(3)$

positive roots for $\mathfrak{k}$: $\{\varepsilon_1 - \varepsilon_2,\ \varepsilon_1 + \varepsilon_2\}$

K-type $(m,\ n) = \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$

$X = S[-2,\ (1,1)]$, the Speh representation of $SL(4,\ \mathbb{R})$ with Knapp-Vogan parameter $m = -2$, lowest K-type $(1,\ 1)$.

Example

(Continued) Associated variety of $S[-2,\ (1,1)] = \overline{\mathcal{O}}$,

where $\mathcal{O} \leftrightarrow \begin{bmatrix} 1 & -i & 0 & 0 \\ -i & -1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \end{bmatrix}$.

$\mathcal{O}$ has height two.

$R[\overline{\mathcal{O}}]^{\mathfrak{n}_\mathbf{c}} = \mathbb{C}[u_1,\ u_2]$

u_1 has degree 1 and weight: $2\left(\frac{\varepsilon_1 - \varepsilon_2}{2}\right) + 2\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) = 2\varepsilon_1$

u_2 has degree 2 and weight: $4\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) = 2\varepsilon_1 + 2\varepsilon_2$

Example

(Conclusion)

K-decomposition of $S[-2,\ (1,1)]$:

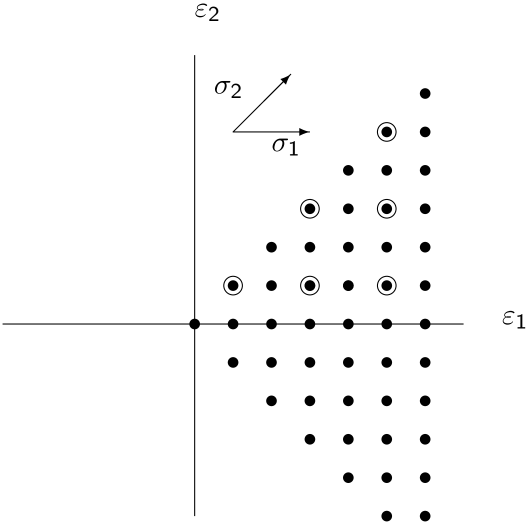
$$S[-2,\ (1,1)] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\ V(2m+1,\ 2n+1),$$

$V(2m+1,\ 2n+1)$ is irr. rep. of $SO(4,\ \mathbf{C})$ with highest weight $(2m+1,\ 2n+1)$.

(See [Knapp-Vogan, 1995], pp586-588)

The diagram below shows the action of σ_1 and σ_2 on highest weights in $S[-2,\ (1,1)]$.

K-type diagram of $S[-2,\ (1,1)]$



Symplectic structure on nilpotent orbits

Ω, an orbit in $N(\mathfrak{g})$ (or $N(\mathfrak{g}_c)$), is a real symplectic manifold.

Kostant-Souriau symplectic form on Ω

For $\Omega = G \cdot E$, the G-invariant form on Ω is defined as:

$w_{\Omega|E}(\bar{Y},\ \bar{Z}) = B(E,\ [Y,\ Z])$,
where B is the Killing form of $\mathfrak{g}$, and $\bar{Y},\ \bar{Z} \in \mathfrak{g}/\mathfrak{g}^E = T_E(\Omega)$.

Example

$\mathfrak{g} = sl(2,\ \mathbb{R})$
$\Omega = G \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x & y - z \\ y + z & -x \end{bmatrix} \middle| x,\ y \in \mathbb{R},\ z = \sqrt{x^2 + y^2} > 0 \right\}$.
Then (up to a constant):

$$w_\Omega = \frac{dx \wedge dy}{z}$$

The moment mapping for nilpotent orbits

For $\xi \in \mathfrak{k}$, define the vector field ξ_Ω.

$$\xi_\Omega f(E') := \frac{df(\exp(-t\xi)\cdot E')}{dt}\Big|_{t=0}, \text{ where } f \in C^\infty(\Omega).$$

The moment map $\Phi_\Omega : \Omega \to \mathfrak{k}^*$ satisfies $d[\Phi_\Omega(\cdot)(\xi)] = -\iota(\xi_\Omega)(w_\Omega)$, $\forall \xi \in \mathfrak{k}$.

In this case, $\Phi_\Omega(E') = B(E'_k, \ \cdot)$. So think of Φ as projection onto $\mathfrak{k}$, i.e., we write, $\Phi_\Omega(E') = E'_k$.

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- Φ is K-equivariant and proper.
- Ω is said to be a Hamiltonian K-space.

Example

For $\mathfrak{g} = sl(2, \mathbb{R})$, $K = SO(2)$ acts on the cone:
$\Omega = \{(x, y, z) | x, y \in \mathbb{R}, z = \sqrt{x^2 + y^2} > 0\}$.

Identify $\mathfrak{k}^*$ with the z-axis and $\mathfrak{p}^*$ with the xy-plane.

- Φ projects the upper cone onto the positive z-axis.
- K orbits in Ω are circles.
- Φ is "one-to-one" on K-orbits.

In general what can we say about $\Phi(\Omega)$, when Ω is nilpotent?

There are many nice results on the image of the moment map for compact Hamiltonian K-spaces.

Multiplicity-free spaces and Poisson Algebras

Let $(M, \ \rho)$ be a symplectic manifold with a Hamiltonian action by a compact group K with moment map $\Phi : M \to \mathfrak{k}^*$.

Definition

The Hamiltonian K-action on M is said to be multiplicity-free if Φ is one-to-one on K-orbits in M.

$C^\infty(M)$ is a Poisson algebra: $f \in C^\infty(M) \mapsto$ the vector field X_f, such that $df(Y) = \rho(X_f, \ Y)$, $\forall$ smooth vector fields Y on M.

If $f, \ g \in C^\infty(M)$, $\{f, \ g\} := \rho(X_f, \ X_g\}$

M is multiplicity-free $\iff$ $C^\infty(M)^K$ is commutative.

Kostant-Sekiguchi Correspondence

There is is a bijection: $N(\mathfrak{g})/G \leftrightarrow N(\mathfrak{p}_{\mathrm{c}})/K_{\mathrm{c}}$

This bijection can be defined in terms of "Kostant-Sekiguchi" $sl(2)$-triples:

$$\{H,\ E,\ F\} \quad \mapsto \{i(E-F),\ \tfrac{1}{2}(H-i(E+F)),\ \tfrac{1}{2}(H+i(E+F))\}$$
(assuming $\theta(E)=-F$)

$$\{x,\ e,\ f\} \quad \mapsto \{e+f, \tfrac{1}{2}(-ix+i(e-f)),\ \tfrac{1}{2}(ix+i(e-f))\}$$
(assuming $\bar{e}=f$)

Denote the bijection $\Omega \mapsto \mathcal{O}_\Omega$. $(\Omega,\ \mathcal{O})$ will be said to be a *KS* pair if $\mathcal{O}=\mathcal{O}_\Omega$.

Some properties of the KS pair $(\Omega,\ \mathcal{O})$

1. $G_{\mathsf{c}} \cdot \Omega = G_{\mathsf{c}} \cdot \mathcal{O}$.
2. (Barbasch-Sepanski) If $(\Omega_i,\ \mathcal{O}_i)$, $i = 1, 2$, are KS pairs then $\Omega_1 \subset \overline{\Omega_2} \iff \mathcal{O}_1 \subset \overline{\mathcal{O}_2}$
3. [Vergne, 1995] There is a K-equivariant diffeomorphism $\Omega \to \mathcal{O}$.

Characterizing multiplicity-free nilpotent orbits

Theorem

[K, 2002] If $(\Omega, \mathcal{O})$ is a KS pair, Ω is multiplicity free $\iff$ $\mathcal{O}$ is spherical.

Characterizing multiplicity-free nilpotent orbits

Theorem

[K, 2002] If $(\Omega, \mathcal{O})$ *is a KS pair,* Ω *is multiplicity free* $\iff$ $\mathcal{O}$ *is spherical.*

"Multiplicity-freeness" on the symplectic side of the KS correspondence coincides with "multiplicity-freeness" on the algebraic side of the correspondence.

An "enhanced" KS-correspondence

Suppose $(\Omega,\ \mathcal{O})$ is a Kostant-Sekiguchi pair.

- Many properties of (the K-action on) Ω are mirrored in properties of (the $K_{\mathbf{c}}$-action on) $\mathcal{O}$.
- This suggests that we develop an "enhanced" version of the Kostant-Sekiguchi correspondence.

Here is an attempt to formulate such an enhancement.

The Kostant-Sekiguchi Ansatz

Ansatz

There is a matching between the "interesting" K-invariants arising from the Hamiltonian K-space structure of Ω and the "interesting" $K_{\mathbf{c}}$-invariants arising from the $K_{\mathbf{c}}$-action on $\mathcal{O}$.

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Remark

"interesting" $=$ interesting from the point of representation theory.

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Remark

"interesting" = interesting from the point of representation theory.

Think of the Matsuki Correspondence and its extensions.

Complexity of nilpotent orbits

Definition

$c := c_{K_\mathsf{c}}(\mathcal{O})$, the complexity of $\mathcal{O}$, is the co-dimension of a generic orbit of a Borel subgroup of K_c in $\mathcal{O}$.

Complexity of nilpotent orbits

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$c := c_{K_{\mathbb{C}}}(\mathcal{O})$, the complexity of $\mathcal{O}$, is the co-dimension of a generic orbit of a Borel subgroup of $K_{\mathbb{C}}$ in $\mathcal{O}$.

Remark

- *c is the smallest non-negative integer such that if $\lambda \in \widehat{K}$ and $m_{\mathcal{O}}(\lambda) \neq 0$, $m_{\mathcal{O}}(n\lambda)$ grows no faster than a constant multiple of n^c as $n \to \infty$.*
- $c(\mathcal{O}) = c(\overline{\mathcal{O}})$
- $c(\mathcal{O}) = 0 \iff \mathcal{O}$ *is spherical.*

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Example

If $\mathcal{O}$ is the principal K_c-nilpotent in $\mathfrak{p}_\mathrm{c}$ for $\mathfrak{g} = sl(3, \mathbb{R})$, then $c(\mathcal{O}) = 1$.

Example

(continued)

$$R[\overline{\mathcal{O}}] \simeq \sum_{j=0}^{\infty} m_j H_j,$$

where

$$m_j = \begin{cases} \frac{j-1}{2} & \text{if } j \text{ is odd,} \\ \frac{j}{2} + 1 & \text{if } j \text{ is even.} \end{cases}$$

Characterizing complexity of nilpotent orbits

Theorem

[K, 2005] Let $(\Omega,\ \mathcal{O})$ be a KS pair and $\mathcal{A}(\Omega)^K$ denote the Poisson algebra of K-invariant real analytic functions on Ω. Then $c(\mathcal{O})$ measures the "difference" between $\mathcal{A}(\Omega)^K$ and its (Poisson) center i.e., the failure of $\mathcal{A}(\Omega)^K$ to be commutative.

More precisely, we can find $2c(\mathcal{O})$ functionally independent K-invariant real analytic functions on Ω which lie outside the center of $\mathcal{A}(\Omega)^K$ and satisfy one other technical condition. $2c(\mathcal{O})$ is the largest integer for which this is possible.

Some open problems

1. (Convexity) The image of the moment map $\Phi : \Omega \longrightarrow \mathfrak{k}^*$
2. (Duistermaat-Heckman measure) The pushforward to $\mathfrak{k}^*$ of Liouville measure on Ω under Φ

- Fix the KS pair $(\Omega,\ \mathcal{O})$.
- Choose a closed Weyl chamber $\mathfrak{t}^+$ in $\mathfrak{t}^* = \mathit{Hom}(\mathfrak{t},\ \mathbb{R})$.
- $\Gamma(\overline{\mathcal{O}}) \subset i\mathfrak{t}^+$ and $\mathcal{C}(\overline{\mathcal{O}}) \subset i\mathfrak{t}^+$

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Conjecture

$$\overline{\Phi_\Omega(\Omega) \cap \mathfrak{t}^+} = -i\ \left(\mathcal{C}(\overline{\mathcal{O}})\right)$$

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Conjecture

$$\overline{\Phi_\Omega(\Omega) \cap \mathfrak{t}^+} = -i \ \ (\mathcal{C}(\overline{\mathcal{O}}))$$

Remark

The conjecture is equivalent to:

$$-i\,Interior\,(\mathcal{C}(\overline{\mathcal{O}})) \subseteq \Phi_\Omega(\Omega) \cap \mathfrak{t}^+ \subseteq -i \ \ (\mathcal{C}(\overline{\mathcal{O}}) \setminus 0)$$

Some evidence for the conjecture on $\Phi(\Omega)$

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1. Knop (2002) "Convexity of Hamiltonian manifolds"

   Φ proper $\Rightarrow$ Ω is convex (in Knop's sense)

   $\Rightarrow \Phi_\Omega(\Omega) \cap \mathfrak{t}^+$ is locally closed and is locally a polyhedral cone.

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2. Minimal non-zero orbits

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2. Minimal non-zero orbits

3. Other small spherical nilpotent orbits for $\mathfrak{g}$ of low rank

Verifying the conjecture for minimal nilpotent orbits

$\Phi_\Omega(\Omega) \cap \mathfrak{t}^+$ and $\mathcal{C}(\overline{\mathcal{O}})$ are one dimensional:

- $G = KAN$
- Assume K is semi-simple, then Ω_{min} is unique.
- $\Omega_{min} = G \cdot E$, where $N \cdot E = E$ and $A \cdot E = \mathbb{R}^+ E$
- $\Phi(\Omega_{min}) = \Phi(K \cdot \mathbb{R}^+ E) = K \cdot \mathbb{R}^+ \Phi(E)$
- $$R[\overline{\mathcal{O}_{min}}] = \sum_{n=0}^{\infty} V(n\lambda)$$

  , where $\lambda \in K \cdot i\mathbb{R}^+ \Phi(E)$.

For other small spherical nilpotent orbits $G \cdot E$ has a similar decomposition as $K\mathfrak{a}(s) \cdot E$, where $a(s)$ is a certain abelian subspace of $\mathfrak{p}$ (depending on the special KS triple containing E). E is a sum of eigenvectors for $\mathfrak{a}(s)$. This allows one to compute $\Phi(\Omega)$.

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Example

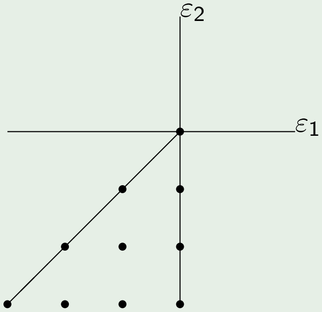
$\mathfrak{g} = sp(4, \ \mathbb{R})$

- $\Omega_1 \leftrightarrow \begin{array}{|c|c|} \hline + & - \\ \hline + & - \\ \hline \end{array}$ ("holomorphic" nilpotent)
- $\Omega_2 \leftrightarrow \begin{array}{|c|c|} \hline + & - \\ \hline - & + \\ \hline \end{array}$ ("pseudo-holomorphic" nilpotent)

As usual $\Delta^+(\mathfrak{k}, \ \mathfrak{t}) = \{\varepsilon_1 - \varepsilon_2\}$

Example

(Continued)

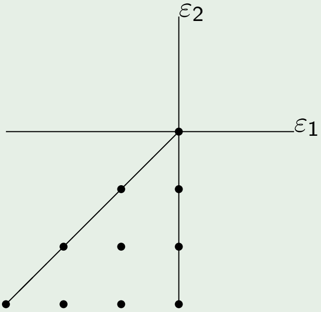


$\Gamma(\overline{\mathcal{O}_1})$ is the span over $\mathbb{Z}_+$ of $-2\varepsilon_2$ and $-2\varepsilon_1 - 2\varepsilon_2$.

$\Phi(\Omega_1) \cap \mathfrak{t}^+$ for $\mathfrak{g} = sp(4, \mathbb{R})$

Example

(Continued)



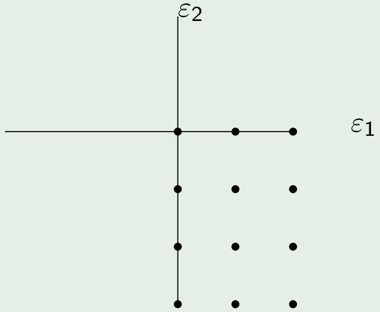
$\Gamma(\overline{\mathcal{O}_1})$ is the span over $\mathbb{Z}_+$ of $-2\varepsilon_2$ and $-2\varepsilon_1 - 2\varepsilon_2$.

$\Phi(\Omega_1) \cap \mathfrak{t}^+$ is ($-i$ times) the half-quadrant: $\varepsilon_2 \leq \varepsilon_1 < 0$.

$\Phi(\Omega_2) \cap \mathfrak{t}^+$ for $\mathfrak{g} = sp(4, \mathbb{R})$

Example

(Conclusion)

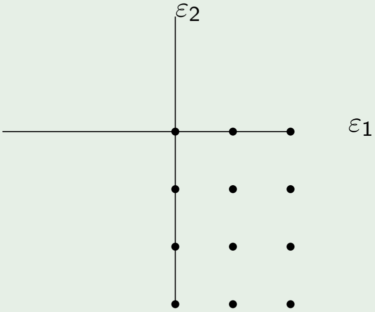


$\Gamma(\overline{\mathcal{O}_2})$ is the span over $\mathbb{Z}_+$ of $2\varepsilon_1$ and $-2\varepsilon_2$.

$\Phi(\Omega_2) \cap \mathfrak{t}^+$ for $\mathfrak{g} = sp(4, \mathbb{R})$

Example

(Conclusion)



$\Gamma(\overline{\mathcal{O}_2})$ is the span over $\mathbb{Z}_+$ of $2\varepsilon_1$ and $-2\varepsilon_2$.

$\Phi(\Omega_2) \cap \mathfrak{t}^+$ is ($-i$ times) the quadrant: $\varepsilon_1 > 0, \ \varepsilon_2 < 0$.

Find a precise description of $\Phi(\Omega) \cap \mathfrak{t}^+$.

$\Phi_\Omega(\Omega) \cap \mathfrak{t}^+$ may have some interesting geometry.

$\Phi_*(\beta_\Omega)$, the pushforward to $\mathfrak{k}^*$ of Liouville measure on Ω

Definition

- $w =$ Kostant-Souriau form on Ω, $d = \frac{\dim \Omega}{2}$.

$$\textit{Liouville measure: } \beta_\Omega = \frac{w^d}{(2\pi)^d d!}$$

- $\Phi_*(\beta_\Omega)(f) = \beta_\Omega(f \circ \Phi)$, for $f \in C_c^\infty(\mathfrak{k}^*)$

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Example

$\mathfrak{g} = sl(2, \mathbb{R})$
$\Omega = \{(x, y, z) | x, y \in \mathbb{R}, z = \sqrt{x^2 + y^2} > 0\}$. $\beta_\Omega = \frac{1}{2\pi} \frac{dx \wedge dy}{z}$.

$\Phi_*(\beta_\Omega) =$ Lebesgue measure on positive half of $\mathbb{R}$.

$M_{\overline{\mathcal{O}}}$, asymptotic K-multiplicity distribution of $\overline{\mathcal{O}}$

Definition

- For $\mu \in \widehat{K}$, set $j_\mu =$ Liouville measure on $K \cdot -i(\mu + \rho)$.
- For $f \in C_c^\infty(\mathfrak{k}^*)$, and $t > 0$, set $f_t(x) = t^{-\dim \mathfrak{k}} f(t^{-1}x)$.
- Set $p = d - \dim k$,

$$M_{\overline{\mathcal{O}}}(f) := \lim_{t \to \infty} t^{-p} \sum_{\mu \in \widehat{K}} m_{\overline{\mathcal{O}}}(\mu) j_\mu(f_t)$$

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Example

$\mathfrak{g} = sl(2,\ \mathbb{R})$ (Continued)

$M_{\overline{\mathcal{O}}}(f) = \lim_{t\to\infty} t^{-1} \sum_{n=1}^{\infty} \delta_n(f_t) =$ Lebesgue measure on positive half of $\mathbb{R}$.

Conjecture

(Vogan, 199?) $M_{\overline{\mathcal{O}}} = \Phi_*(\beta_\Omega)$

Conjecture

(Vogan, 199?) $M_{\overline{\mathcal{O}}} = \Phi_*(\beta_\Omega)$

Status of Conjecture

- Vergne proved the conjecture for complex $\mathfrak{g}$ ([Vergne, 1998]
- (K,) Conjecture holds for many nilpotent orbits (e.g., minimal, even)

Conclusion

Find more interesting "matching" invariants for Ω and $\mathcal{O}$.

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