Singularities of slices in the nilpotent cone

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Aims

General aim

Understand the singularities of nilpotent cones.

More specifically

- describe the singularities of one nilpotent orbit with respect to another orbit in its closure
- case of minimal degenerations: orbits are adjacent in partial order
- case of special nilpotent orbits: the duality of Kraft-Procesi from GL_n has an extension to all types
- case of special pieces: there is an analog of Lusztig's conjecture on special pieces for slices

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Nilpotent orbit closures

$\mathfrak{g}$ a complex simple Lie algebra (e.g. $\mathfrak{sl}_n$).
G the adjoint Lie group of $\mathfrak{g}$

$x \in \mathfrak{g}$ a nilpotent element, $\mathcal{O}_x := G \cdot x$, the nilpotent orbit of x.

- The closure $\overline{\mathcal{O}}_x$ is a union of finitely many nilpotent orbits.
- $\overline{\mathcal{O}}_x$ is smooth at any point of $\mathcal{O}_x$, but it's singular at any other point (Kaledin).

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Hasse diagram

For two orbits $\mathcal{O}, \mathcal{O}'$,
denote by $\mathcal{O}' < \mathcal{O}$ if $\mathcal{O}' \subsetneq \overline{\mathcal{O}}$.

$0 < \mathcal{O}_{min} < \cdots \textit{could be very complicated} \cdots < \mathcal{O}_{subreg} < \mathcal{O}_{reg}$

- Unique maximal nilpotent orbit $\mathcal{O}_{\text{reg}}$ of dimension equal to the number of roots. Called the regular or principal nilpotent orbit.
- Unique orbit $\mathcal{O}_{\text{subreg}}$ containing all non-regular orbits in its closure. It is of codimension 2 in $\overline{\mathcal{O}}_{reg}$. Called the subregular nilpotent orbit.
- $\mathcal{O}_{min}$ is the minimal non-zero nilpotent orbit. Its closure $\bar{\mathcal{O}}_{min}$ is normal with an isolated singularity at 0. It is the orbit of a highest root vector.

In G_2, the Hasse diagram is

$$0 - A_1 - \tilde{A}_1 - G_2(a_1) - G_2$$

Transverse Slices

Given $\mathcal{O}' < \mathcal{O}$. Let $e \in \mathcal{O}'$.

Definition

A *transverse slice* of $\overline{\mathcal{O}}$ at the point e is a locally closed subvariety $S \subset \overline{\mathcal{O}}$ such that $e \in S$ and the map

$$G \times S \to \overline{\mathcal{O}}$$

is smooth at $(1, e)$.

If S is a slice, then $S \cap \mathcal{O}' = e$ (and reduced).

As we'll see in moment, slices always exist in the present situation.

Smooth Equivalence

We say two singularities $x \in X$ and $y \in Y$ are *smoothly equivalent* if there exists a variety Z and two morphisms

$$(X, x) \xleftarrow{\varphi} (Z, z) \xrightarrow{\psi} (Y, y)$$

with φ and ψ smooth at z.

We then write $\mathrm{Sing}(X, x) = \mathrm{Sing}(Y, y)$.

Smooth equivalence preserves various (analytic) properties at x:

smoothness, normality, unibranchness, rational singularities

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Slices give smooth equivalence

Given a transverse slice $S \subset \bar{\mathcal{O}}$ at e, we have

$$\mathrm{Sing}(S, e) = \mathrm{Sing}(\bar{\mathcal{O}}, e).$$

Slodowy Slices

There exists a natural slice, called the Slodowy slice.

Given nilpotent e, one can find an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$, meaning

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f$$

We denote by $\mathfrak{g}^f$ the centralizer of f in $\mathfrak{g}$. Then the affine subspace

$$S_e := e + \mathfrak{g}^f$$

is a transverse slice in $\mathfrak{g}$ at the point e.

And $S_e \cap \overline{\mathcal{O}}$ is a transverse slice in $\overline{\mathcal{O}}$ at e.

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Extra action on slice

The reductive centralizer $Z := G^{\langle e,h,f \rangle}$ acts on $S_e = e + \mathfrak{g}^f$, and also on $S_e \cap \overline{\mathcal{O}}$.

In general, $S_e \cap \overline{\mathcal{O}}$ is not irreducible (always connected).

So the component group $A(e) := Z/Z^0$ will act on the irreducible components.

Goal

Identify $\mathrm{Sing}(S_e \cap \bar{\mathcal{O}}, e)$ together with the action of $A(e)$

Two types of singularities

For now, let $\mathcal{O}' < \mathcal{O}$ be adjacent in the partial order. Called a **minimal degeneration**.

In this case, the variety $S_e \cap \bar{\mathcal{O}}$ has an isolated singularity at e.

There are two families of singularities that arise in all but a finite number of cases.

- A-D-E or Du Val singularities (in the case of dimension two).
- Those associated to minimal nilpotent orbits (in the case of dimension ≥ 4.)

Singularities coming from minimal orbits

For each simple Lie algebra, we refer to the singularity of $\overline{\mathcal{O}}_{min}$ at 0 according to lower case letters:

$$a_n, b_n, c_n, d_n, g_2, f_4, e_6, e_7, e_8.$$

Some of these varieties carry natural actions which fix 0, coming from outer automorphisms of $\mathfrak{g}$.
We write these as:

$$a_n^+, d_n^+, d_4^{++}, e_6^+,$$

where d_4^{++} refers to the S_3 action.

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A-D-E singularities

Take a finite subgroup $\Gamma \subset SL_2(\mathbb{C})$

Define $X = \mathbb{C}^2/\Gamma$, the quotient.

The functions on X are the invariant functions $\mathbb{C}[x, y]^\Gamma$.

X is two-dimensional, normal, and has an isolated singularity at 0.

Which Γ are possible?

$$\begin{array}{ccccc} & & \{\pm I\} & & \\ & & \downarrow & & \\ \Gamma & \subset & SU(2) & \subset & SL_2(\mathbb{C}) \\ \downarrow & & \downarrow & & \\ \Gamma' & \subset & SO(3) & & \end{array}$$

Γ' is the (rotational) symmetry group of a polyhedron.

- elongated pyramid $\rightarrow$ cyclic group of order n
- elongated double pyramid $\rightarrow$ dihedral group of order $2n$
- tetrahedron, cube, icosahedron $\rightarrow$ Alt_4, S_4, Alt_5.

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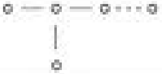
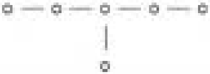
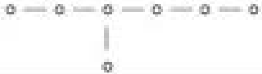
Table of A-D-E singularities

Name	Equation	Group	Resolution graph
A_n	$x^2 + y^2 + z^{n+1}$	cyclic $\mathbb{Z}/(n+1)$	
D_n	$x^2 + y^2 z + z^{n-1}$	binary dihedral $\mathrm{BD}_{4(n-2)}$	
E_6	$x^2 + y^3 + z^4$	binary tetrahedral	
E_7	$x^2 + y^3 + yz^3$	binary octahedral	
E_8	$x^2 + y^3 + z^5$	binary icosahedral	

Automorphisms of the A-D-E surfaces

Part of the McKay correspondence story

These surfaces X also have automorphisms corresponding to the automorphisms of the Dynkin diagram. Corresponds to permuting the $\mathbb{P}^1$'s.

From the perspective of the finite group Γ, the automorphisms are induced from the normalizer of Γ in SL_2.

For example, $V_4 = \Gamma'_{D_4}$ is normal in $S_4 = \Gamma'_{E_7}$ and the quotient is S_3.
This gives triality for D_4.

Notation for A-D-E surfaces

Refer to singularity of type X according to the type of its Dynkin diagram:

$$A_n, D_n, E_6, E_7, E_8$$

and accounting for the automorphisms:

$$\begin{array}{lcl} B_n = A^+_{2n-1} & := & A_{2n-1} \text{ with } S_2\text{-action} \\ C_n = D^+_{n+1} & := & D_{n+1} \text{ with } S_2\text{-action} \\ F_4 = E^+_6 & := & E_6 \text{ with } S_2\text{-action} \\ G_2 = D^{++}_4 & := & D_4 \text{ with } S_3\text{-action} \end{array}$$

Note: $A_1 = a_1$.

How do these arise as singularities of slices?

Theorems of Brieskorn and Slodowy

Theorem (Brieskorn, 1970)

Suppose that $\mathfrak{g}$ is of ADE type Γ. Then

$$\mathrm{Sing}(\overline{\mathcal{O}}_{\mathrm{reg}}, \mathcal{O}_{\mathrm{subreg}}) = \Gamma$$

is an ADE singularity of the same type.

Slodowy explained what happens for non simply-laced Lie algebras:

$$\begin{aligned} B_n &= A_{2n-1}^{+} \\ C_n &= D_{n+1}^{+} \\ F_4 &= E_6^{+} \\ G_2 &= D_4^{++} \end{aligned}$$

and the action is the one coming from $A(e)$.

Results of Kraft and Procesi for $\mathfrak{sl}_n$

Theorem (Kraft-Procesi, case of $\mathfrak{sl}_n$)

If the r first lines and the s first columns of λ et μ are identical, and if $\hat{\lambda}$ and $\hat{\mu}$ are the partitions obtained by removing those common lines and columns, then

$$\mathrm{Sing}(\overline{\mathcal{O}}_\lambda, \mathcal{O}_\mu) = \mathrm{Sing}(\overline{\mathcal{O}}_{\hat{\lambda}}, \mathcal{O}_{\hat{\mu}})$$

Eventually we get down to one of the two cases for a smaller rank $\mathfrak{sl}$.

It follows that all minimal degenerations are either:

- A_k (codimension 2)
- a_k (codimension > 2)

The two cases are perfectly exchanged a duality...

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Example: type A_5

$$
\begin{tikzcd}[column sep=small]
 & (\mathbf{6}) \arrow[d, dash, "A_5"] & & & 30 \\
 & (\mathbf{5},\mathbf{1}) \arrow[d, dash, "A_3"] & & & 28 \\
 & (\mathbf{4},\mathbf{2}) \arrow[dl, dash, "A_1"] \arrow[dr, dash, "A_1"'] & & & 26 \\
(\mathbf{4},\mathbf{1}^{\mathbf{2}}) \arrow[dr, dash, "A_2"'] & & (\mathbf{3}^{\mathbf{2}}) \arrow[dl, dash, "A_2"] & & 24 \\
 & (\mathbf{3},\mathbf{2},\mathbf{1}) \arrow[dl, dash, "a_2"] \arrow[dr, dash, "a_2"'] & & & 22 \\
(\mathbf{3},\mathbf{1}^{\mathbf{3}}) \arrow[dr, dash, "a_1"'] & & (\mathbf{2}^{\mathbf{3}}) \arrow[dl, dash, "a_1"] & & 18 \\
 & (\mathbf{2}^{\mathbf{2}},\mathbf{1}^{\mathbf{2}}) \arrow[d, dash, "a_3"] & & & 16 \\
 & (\mathbf{2},\mathbf{1}^{\mathbf{4}}) \arrow[d, dash, "a_5"] & & & 10 \\
 & (\mathbf{1}^{\mathbf{6}}) & & & 0
\end{tikzcd}
$$

Other classical groups

Kraft-Procesi (1982)

In other classical types, if $\mathcal{O}' < \mathcal{O}$ a minimal degeneration, then $\mathrm{Sing}(\overline{\mathcal{O}}, \mathcal{O}')$ is equivalent to one of the following:

$$A_{2k-1}$$
$$D_k$$
$$A_{2k-1} \cup A_{2k-1}$$

$$b_k$$
$$c_k$$
$$d_k$$

Bring in $A(e)$-action

Proposition (FJLS, 2012)

In the classical groups B_n, C_n, D_n, the component group $A(e)$ acts as transitively as it possible can on irreducible components of minimal degenerations.

We must work in the full orthogonal group in type D.

The minimal degenerations with $A(e)$-action are:

$$
\begin{aligned}
&B_k := A^+_{2k-1}\\
&C_k := D^+_{k+1}\\
&B_k \cup B_k\\
&b_k\\
&c_k\\
&d^+_k
\end{aligned}
$$

$A(e)$ permutes the irreducible components in the $B_k \cup B_k$ cases.

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Example

In C_6, which is Sp_{12}, orbits correspond to certain partitions of 12.

- $\mathcal{O}$ corresponds to $(5, 5, 1, 1)$
- $e \in \mathcal{O}'$ corresponds to $(4, 4, 2, 2)$
- Kraft-Procesi say that the singularity is $A_3 \cup A_3$.
- $A(e) = S_2 \times S_2$.
- one generator of $A(e)$ flips the components, the other acts non-trivially on both components at the same time.

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Other known results

Kraft 1989 Case of G_2.

New singularity, called m, for the singularity of $\tilde{A}_1$ along A_1.

$$\begin{array}{ccc} \mathbb{C}^2 & \to & \mathbb{C}^7 = \mathbb{C}^3 \oplus \mathbb{C}^4 \\ (u,v) & \mapsto & (u^2, uv, v^2; \quad u^3, u^2v, uv^2, v^3) \end{array}$$

The normalization of m is smooth. The normalization is just $\mathbb{C}^2$.

Summary of our results in exceptional groups

Theorem: In the exceptional groups, the irreducible components of minimal degenerations are:

- A-D-E type
- minimal orbit closure (predicted by reductive centralizer, which acts transitively on smooth part)
- m
- $\mathrm{Sing}(A_3 + 2A_1, 2A_2 + 2A_1)$ in E_8, of dimension 4. It carries action of Sp_4, non-normal but unibranched. Its normalization is $\mathbb{C}^4$.
- Three distinct cases of dimension 4, that carry an action of SL_2, but **not** transitive on the smooth part:
  - $\mathrm{Sing}(2A_2 + A_1, A_2 + 2A_1)$ in E_6
  - $\mathrm{Sing}(A_4 + A_1, A_3 + A_2 + A_1)$ in E_7
  - $\mathrm{Sing}(A_4 + A_3, A_4 + A_2 + A_1)$ in E_8

Also: $A(e)$ acts transitively on irreducible components of slice

Normality of orbit closures

The singularity $B_k \cup B_k =: 2.B_k$ shows that the orbit $\overline{\mathcal{O}}$ is not unibranched at $e \in \mathcal{O}'$. In particular, $\overline{\mathcal{O}}$ is not normal at e.

A consequence of the Kraft-Procesi work is that in the classical groups is that this is the only way for $\overline{\mathcal{O}}$ to be not normal.

That is, $\overline{\mathcal{O}}$ fails to be normal if and only if it if it fails to be normal in codimension two and and the singularity is $B_k \cup B_k$.

In the exceptional groups, normality can fail in three additional ways (and these should conjecturally be all of them):

- it is branched at a minimal degeneration (e.g. $3.a_1$ or $2.g_2$).
- it is branched at a point farther down (detected by Green functions).
- the singularity m arises (non-normal and unibranched).
- the one case in E_8 (non-normal and unibranched).

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Idea of proof

If $\dim S = 2$, general results say S has an isolated symplectic singularity at e and therefore, up to normalization, it is of type $A - D - E$.

Then we can locate a minimal resolution, similar to the Springer resolution of the whole nilpotent cone, and then use a result of Borho-MacPherson to count the number of $\mathbb{P}^1$'s over e to determine which A, D, E singularity occurs. This invokes using the Springer correspondence and looking up tables of Green functions.

If $\dim S \geq 4$, then the singularity is closely related to the minimal orbit in a reductive centralizer of e. Many times, in fact, S is exactly $e +$ this minimal orbit.

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Special nilpotent orbits

Springer correspondence:
W= Weyl Group.

$$\{irreducible \quad W-modules\} \hookrightarrow$$

$$\{(\mathcal{O},\phi) | \mathcal{O} \quad nilp. \quad orbit, \phi \quad irred \quad representation \quad of \quad A(\mathcal{O})\}.$$

A nilp. orbit $\mathcal{O}$ is called **special** if the irred. rep. $\rho_{(\mathcal{O},1)}$ is a special W-representation.

Special nilpotent orbits play a key role in several problems in representation theory:

- Classification of irred. complex rep. of a reductive gp over a finite field
- classification of primitive ideals in the enveloping alg. of a semi-simple Lie algebra

Special orbits and Lusztig-Spaltenstein involution

- $\mathfrak{g}$ of type A, every nilpotent orbit is special.
- $\mathfrak{g}$ of type B or C, $\mathcal{O}_{\mathbf{p}}$ is special $\Leftrightarrow$ the transpose partition $\mathbf{p}^t$ defines a nilpotent orbit in $\mathfrak{g}$.
- $\mathfrak{g}$ of type D_n, $\mathcal{O}_{\mathbf{p}}$ is special $\Leftrightarrow$ the transpose partition $\mathbf{p}^t$ defines a nilp. orbit in C_n.

If we take the Hasse diagram of *special orbits*, then there exists an order-reversing Lusztig-Spaltenstein involution d.

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Special orbits and Lusztig-Spaltenstein involution

We can also view d as going between special orbits in $\mathfrak{g}$ and those in its Langlands dual ${}^L\mathfrak{g}$.

The involution is given by the map $\mathbf{p} \mapsto \mathbf{p}^t$ in the case of types A, B, C.

Goal

Identify the singularities between adjacent special orbits and what happens under d

Minimal special orbits

We can look at the closure of the minimal non-zero special orbit in each simple Lie algebra. This is a normal variety.

In simply-laced cases, minimal orbit is special.

In the non-simply laced cases, we obtain the varieties:

$$b_n^{sp} := 0 \xleftarrow{b_n} A_1 \xleftarrow{a_1} \tilde{A}_1$$

$$c_n^{sp} := 0 \xleftarrow{c_n} A_1 \xleftarrow{c_{n-1}} \tilde{A}_1$$

$$f_4^{sp} := 0 \xleftarrow{f_4} A_1 \xleftarrow{c_3} \tilde{A}_1$$

$$g_2^{sp} := 0 \xleftarrow{g_2} A_1 \xleftarrow{m} \tilde{A}_1 \xleftarrow{a_1} G_2(a_1)$$

Minimal special orbits

These can also be viewed as taking a minimal orbit in a simply-laced Lie algebra and using the outer automorphism.

$$b_n^{sp} = d_{n+1}/S_2$$

$$c_n^{sp} = a_{2n-1}/S_2$$

$$f_4^{sp} = e_6/S_2$$

$$g_2^{sp} = d_4/S_3$$

Theorem, Duality

Let $(\mathcal{O}, \mathcal{O}')$ be a minimal degeneration of two special nilpotent orbits.

- Either $\mathcal{O}' \subset \bar{\mathcal{O}}$ or $d(\mathcal{O}) \subset \overline{d(\mathcal{O}')}$ is of codimension 2. Observed by Lusztig.
- Under a technical condition involving Lusztig's canonical quotient, duality interchanges types, just like Kraft-Procesi observed in GL_n:

  $A_n \leftrightsquigarrow a_n \qquad D_n \leftrightsquigarrow d_n \qquad E_n \leftrightsquigarrow e_n$
  $B_n \leftrightsquigarrow c_n^{sp} \text{ or } a_{2n-1}^{+}$
  $C_n \leftrightsquigarrow b_n^{sp} \text{ or } d_{n+1}^{+}$
  $G_2 \leftrightsquigarrow g_2^{sp} \text{ or } d_4^{++}$
  $F_4 \leftrightsquigarrow f_4^{sp} \text{ or } e_6^{+}$
- The $A(e)$-action is critical. We quotient by part of the $A(e)$-action and the remaining action determines the duality.

Theorem, Duality

Let $(\mathcal{O}, \mathcal{O}')$ be a minimal degeneration of two special nilpotent orbits.

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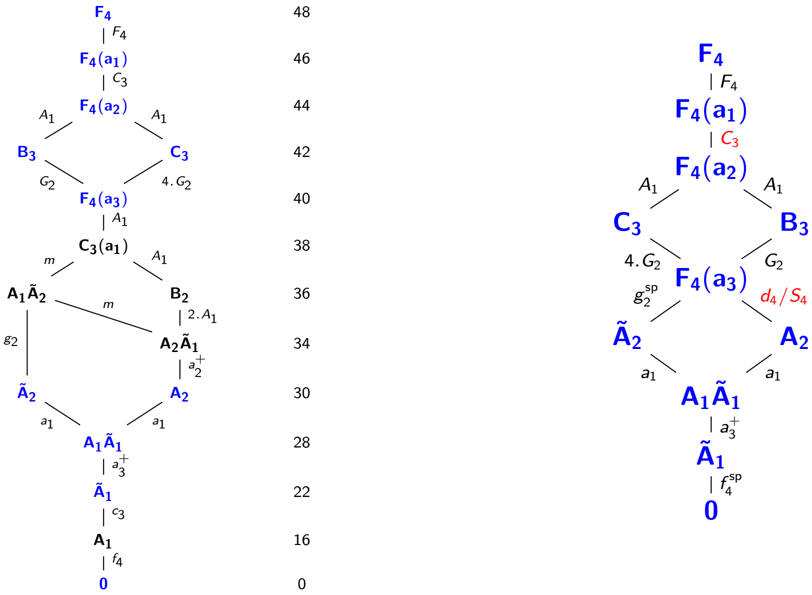
  $G_2 \leftrightsquigarrow g_2^{sp} \text{ or } d_4^{++}$
  $F_4 \leftrightsquigarrow f_4^{sp} \text{ or } e_6^{+}$

- The $A(e)$-action is critical. We quotient by part of the $A(e)$-action and the remaining action determines the duality.

Type F_4



F4
F4
F4(a1)
C3
F4(a2)
A1
A1
B3
C3
G2
4.G2
F4(a3)
A1
C3(a1)
m
A1
A1Ã2
m
B2
2.A1
g2
A2Ã1
a2+
Ã2
A2
a1
a1
A1Ã1
a3+
Ã1
c3
A1
f4
0
48
46
44
42
40
38
36
34
30
28
22
16
0
F4
F4
F4(a1)
C3
F4(a2)
A1
A1
C3
B3
4.G2
G2
F4(a3)
g2sp
d4/S4
Ã2
A2
a1
a1
A1Ã1
a3+
Ã1
f4sp
0

More on the duality

- The duality fails, in some sense, when the $A(e)$-action does not descend to Lusztig's quotient $\bar{A}(e)$. When it does fail:

  In codimension 2, we always have C_k matched up with c_{k-1}.

  Stranger: in codim ≥ 4, we get some new singularities:

$$d_4/S_4 \text{ matched with } G_2$$
$$d_k/(S_2 \times S_2) \text{ matched with } C_{k-1}.$$

- Another beautiful phenomenon: the slice between special orbits (when the duality works) always comes from a smaller Lie algebra, except for the three exceptional orbits in E_7 and E_8 (these are the ones where tensoring with sign didn't quite work). There, the singularities are of type A_2^+, A_4^+ and a_2^+, a_4^+.

Special pieces

Definition

Let $\mathcal{O}$ be a special nilp. orbit. The **special piece** $\mathcal{P}(\mathcal{O})$ *containing $\mathcal{O}$ is the locally-closed subvariety of $\mathfrak{g}$ consisting of nilp. orbits $\mathcal{O}' \subset \bar{\mathcal{O}}$ which is not contained in the closure of any special nilp. orbits $\mathcal{O}_1 \subsetneq \bar{\mathcal{O}}$.*

Lusztig conjectured in 1981 that every special piece is rationally smooth (i.e. a rational homology manifold).

This has been proved by Kraft-Procesi (for classical types), Beynon-Spaltenstein (for E_n), Shoji (for F_4) and Lusztig (for G_2).

A conjecture of Lusztig

In 1997, Lusztig formulated the following conj. to explain the rational smoothness:

Conjecture (Lusztig)

Every special piece $\mathcal{P}(\mathcal{O})$ is a finite quotient of a smooth variety P/H, and the orbits in $\mathcal{P}(\mathcal{O})$ correspond to the images of points in P whose H-stabilizer are conjugate in H.

- Known for classical Lie algebras by Kraft-Procesi (1989).
- For exceptional cases, Lusztig has predicted the group H and the correspondence between conj. classes in H and nilpotent orbits contained in $\mathcal{P}(\mathcal{O})$.

Conjecture (Achar-Sage, 2009)

Every special piece is normal.

Our conjecture on special pieces

Conjecture

Consider a special piece $\mathcal{P}(\mathcal{O})$ in an exceptional Lie algebra and let $\mathcal{O}_m \subset \mathcal{P}(\mathcal{O})$ be a minimal orbit in $\mathcal{P}(\mathcal{O})$. Then the transverse slice in $\bar{\mathcal{O}}$ to $\mathcal{O}_m$ is isomorphic to one of the following

$$(\mathfrak{h}_n \oplus \mathfrak{h}_n^*)^k / \mathcal{S}_{n+1}$$

where $\mathfrak{h}_n$ is a Cartan subalgebra in A_n on which the Weyl group $W(A_n) = \mathcal{S}_{n+1}$ acts and n is an integer between 1 and 4.

This is the analog of Lusztig's conjecture in the setting of slices, since $H = S_{n+1}$.

Special pieces theorem

Theorem

Our conjecture on slices of special pieces holds for all except the following two pieces in E_8 are unknown:
$(E_8(a_7), A_4 + A_3) \rightsquigarrow (\mathfrak{h}_4 \oplus \mathfrak{h}_4^*)/\mathcal{S}_5$
$(D_4(a_1) + A_1, 2A_2 + 2A_1) \rightsquigarrow (\mathfrak{h}_2 \oplus \mathfrak{h}_2^*)^2/\mathcal{S}_3.$

A similar statement is also true in the classical groups, where H is elementary abelian.

Corollary

The normality conjecture of Achar-Sage holds for special pieces except possibly for the above two special pieces.

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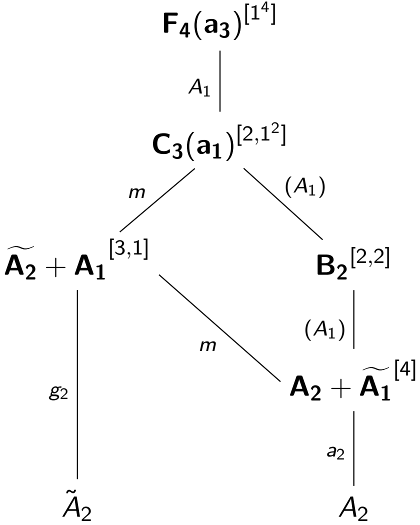
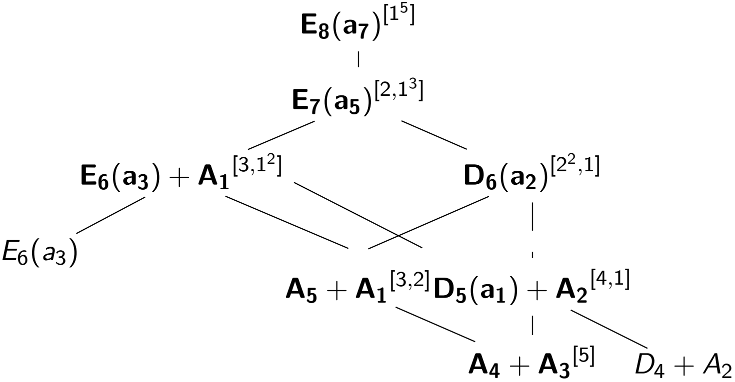

$\mathbf{F_4(a_3)}^{[1^4]}$
A_1
$\mathbf{C_3(a_1)}^{[2,1^2]}$
m
(A_1)
$\widetilde{\mathbf{A_2}} + \mathbf{A_1}^{[3,1]}$
$\mathbf{B_2}^{[2,2]}$
(A_1)
m
g_2
$\mathbf{A_2} + \widetilde{\mathbf{A_1}}^{[4]}$
a_2
$\tilde{A}_2$
A_2


$\mathbf{E_8(a_7)}^{[1^5]}$
$\mathbf{E_7(a_5)}^{[2,1^3]}$
$\mathbf{E_6(a_3)+A_1}^{[3,1^2]}$
$\mathbf{D_6(a_2)}^{[2^2,1]}$
$E_6(a_3)$
$\mathbf{A_5+A_1}^{[3,2]}$
$\mathbf{D_5(a_1)+A_2}^{[4,1]}$
$\mathbf{A_4+A_3}^{[5]}$
D_4+A_2