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Spectral Sequences: some representation-theoretic applications

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Outline:

1. Spectral sequences / a brief tour
2. Proof of a Langlands' conjecture on $L^2(\Gamma\backslash G)$ multiplicities / going well beyond it: a cohomological meaning of multiplicities of Vogan-Zuckerman derived functor modules
3. a BGG type resolution of holomorphic Verma and produced modules for all Hermitian symmetric spaces G/K, and computation of E_1 terms of the associated Grothendieck spectral sequence

Spectral Sequences / a brief tour

A (cohomological) spectral sequence (s.s.) is a family

$$\{ E_r^{pq}, d_r^{pq} \}_{r,p,q \in \mathbb{Z}}$$

[we also write $E_r^{p,q}$, $d_r^{p,q}$]

where $d_r^{pq} : E_r^{pq} \longrightarrow E_r^{p+r,\, q-r+1}$

is a homomorphism of abelian groups [or R-modules / more generally] such that

(a) the composition

$$E_r^{p-r,\, q+r-1} \xrightarrow{d_r^{p-r,\, q+r-1}} E_r^{pq} \xrightarrow{d_r^{pq}} E_r^{p+r,\, q-r+1}$$

is the zero map $\left[\operatorname{Im} d_r^{p-r,\, q+r-1} \subset \ker d_r^{p} \right.$

(b) $E_{r+1}^{pq} \simeq \ker d_r^{pq} \big/ \operatorname{Im} d_r^{p-r,\, q+r-1}$

Main Example: the spectral sequence of a filtered complex

Let $C = \{C^n, d_n\}_{n\in\mathbb{Z}}$ be a

cochain complex of abelian groups:

$C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1}$ is the zero map,

$H^n(C) \overset{\text{def.}}{=} \ker d_n / \operatorname{Im} d_{n-1}$ = n^{th}-dim. cohomology of C

Assume each C^n has a decreasing filtration: $\exists$ a family $\{F_pC^n\}_{p\in\mathbb{Z}}$

$\ni$ $F_pC^n \overset{(i)}{\subset} C^n$ is a subgroup,

$F_{p+1}C^n \overset{(ii)}{\subset} F_pC^n$, $d_nF_pC^n \overset{(iii)}{\subset} F_pC^{n+1}$

Then we can define a s.s.

$\{E_r^{pq}, d_r^{pq}\}_{r,p,q\in\mathbb{Z}}$ as follows:

First define abelian groups

$$Z_r^{pq} = \{ x \in F_p C^{p+q} \mid d_{p+q} x \in F_{p+r} C^{p+q+1} \}$$

$$B_r^{pq} = d_{p+q-1} Z_r^{p-r,\, q+r-1} \equiv d_{p+q-1} F_{p-r} C^{p+q-1} \cap F_p C^{p+}$$

$$\Rightarrow B_{r-1}^{pq},\ Z_{r-1}^{p+1,\, q-1} \subset Z_r^{pq} \text{ are subgroups.}$$

$$E_r^{pq} \overset{\text{def.}}{=} Z_r^{pq} \big/ \left(B_{r-1}^{pq} + Z_{r-1}^{p+1,\, q-1} \right)$$

Secondly : note that $d_{p+q} : Z_r^{pq} \longrightarrow Z_r^{p+r,\, q-r+1}$,

$d_{p+q} : B_{r-1}^{pq} \overset{\text{def.}}{=} d_{p+q-1} Z_{r-1}^{p-r+1,\, q+r-2} \longrightarrow 0$ $\left[\text{since } d_{n-1} \circ d_n \overset{\text{def.}}{=} 0 \right]$

, and $B_{r-1}^{p+r,\, q-r+1} \overset{\text{def.}}{=} d_{p+q} Z_{r-1}^{p+1,\, q-1} \Longrightarrow d_{p+q} : Z_{r-1}^{p+1,\, q-1}$

$\longrightarrow B_{r-1}^{p+r,\, q-r+1}$ which means that

d_{p+q} <u>induces</u> <u>a</u> <u>quotient</u> <u>map</u>

$$d_r^{pq} : E_r^{pq} \overset{\text{def.}}{=} Z_r^{pq} \big/ \left(B_{r-1}^{pq} + Z_{r-1}^{p+1,\, q-1} \right) \longrightarrow$$

$$E_r^{p+r,\, q-r+1} \overset{\text{def.}}{=} Z_r^{p+r,\, q-r+1} \big/ \left(B_{r-1}^{p+r,\, q-r+1} + Z_{r-1}^{p+r+1,\, q-r} \right)$$

One has $\boxed{(a)\quad d_r^{pq} \circ d_r^{p-r,\, q+r-1} = 0}$

since (again) $d_n \circ d_{n-1} = 0 \Longrightarrow d_{p+q} \circ d_{p+q-1} = 0$

Regarding $\boxed{(b)\quad E_{r+1}^{pq} \simeq \ker d_r^{pq} / \operatorname{Im} d_r^{p-r,\, q+r-1}}$

Let $\pi : Z_r^{pq} \longrightarrow E_r^{pq} \overset{\text{def.}}{=} Z_r^{pq} / (B_{r-1}^{pq} + Z_{r-1}^{p+q,\, q-1})$

be the natural map. Then

$\pi(Z_{r+1}^{pq}) = \ker d_r^{pq}$ and an isomorphism $\phi : E_{r+1}^{pq} \xrightarrow{\text{onto}} \ker d_r^{pq} / \operatorname{Im} d_r^{p-r,\, q+r-1}$

is given by $\boxed{\phi(x + B_r^{pq} + Z_r^{p+q,\, q-1}) \overset{\text{def.}}{=} \pi x + \operatorname{Im} d_r^{p-r,\, q+r-1}}$

$\boxed{\text{for } x \in Z_{r+1}^{pq}}$.

Regularity

To tightly connect the s.s. with the cohomology $H^n(C)$ we assume the **regularity conditions**:

(i) $\forall n$ $\exists$ some $s(n) \in \mathbb{Z} \ni F_p C^n = 0$ for $p > s(n)$ [often $s(n) = n$]

(ii) $F_p C^n = C^n$ $\forall n$ when $p \leq 0$.

Then the s.s. converges to an Abelian group E_∞^{pq}; i.e. $\forall p, q$ $\exists$ $r(p,q) \in \mathbb{Z}$ $\ni E_r^{pq} \simeq E_\infty^{pq}$ for $r > r(p,q)$.

In fact: Let $\pi_n : \ker d_n \to H^n(C) \overset{\text{def.}}{=} \dfrac{\ker d_n}{\operatorname{Im} d_{n-1}}$

so $\{ F_p H^n(C) \overset{\text{def.}}{=} \pi_n(F_p C^n \cap \ker d_n) \}_{p \in \mathbb{Z}}$ is a decreasing filtration of $H^n(C)$.

Then $\boxed{E_\infty^{pq} \simeq F_p H^{p+q}(C) \big/ F_{p+1} H^{p+q}(C)}$

and $\boxed{r(p,q) = \max\{p, s(p+q+1) - p\}}$

Consequences of regularity and the description of E_∞^{pq}:

s.s. degeneration at $r=2,1$ and cohomology computation:

(a.) If $r \geq 2$ $\ni$ for some $q_0 \in \mathbb{Z}$

$E_r^{pq} = 0 \quad \forall p, \forall q \neq q_0$, then

$$\boxed{H^p(C) \simeq E_r^{p-q_0, q_0} \quad \forall p}.$$

(b.) If $r \geq 1$ $\ni$ for some $p_0 \in \mathbb{Z}$

$E_r^{pq} = 0 \quad \forall q, \forall p \neq p_0$, then

$$\boxed{H^p(C) \simeq E_r^{p_0, p-p_0} \quad \forall p}.$$

Example : the Borel-Le Potier s.s.

Borel-Le Potier (B-L) diagram :

hol. vector bundle

$$\begin{tikzcd} E \arrow[r] & X \arrow[d, "\pi"'] \\ & Y \end{tikzcd}$$

hol. fibre bundle with compact fibre F

such that E is locally trivial over Y;

i.e. $\exists$ a hol. vector bundle $E_0 \longrightarrow F$ with the property that each $y \in Y$ has an open neigh. U with a hol. trivialization $\phi_U : \pi^{-1}(U) \longrightarrow U \times F$ covered by a hol. vector bundle isomor. $\Psi_U : E_{\pi^{-1}(U)} \longrightarrow U \times E_0$:

$$\begin{tikzcd} E|_{\pi^{-1}(U)} \arrow[r, "\Psi_U"] \arrow[d] & U \times E_0 \arrow[d] \\ \pi^{-1}(U) \arrow[r, "\phi_U"] \arrow[dr, "\pi"'] & U \times F \arrow[d] \\ & U \end{tikzcd}$$

commutes .

$$\boxed{\begin{tikzcd} E|_{\pi(U)} \overset{\text{def.}}{=} i_U^* E \arrow[r] \arrow[d] & E \arrow[d] \\ i_U : \pi^{-1}(U) \arrow[r, "\text{inclusion}"] & X \end{tikzcd}}$$

Form the <u>Dolbeault</u> <u>cohomology</u> <u>bundle</u> <u>of</u> E, <u>of</u> <u>type</u> (p,q) :

$$\mathbb{H}^{p,q}(E) \overset{\text{def.}}{=} \bigcup_{y\in Y} H^{p,q}(X_y, E|_{X_y}) \times \{y\}$$

[for $E|_{X_y} \longrightarrow X_y \overset{\text{def.}}{=} \pi^{-1}(y)$ = fibre over y

$\therefore X_y \simeq F$ compact $\Rightarrow \dim H^{p,q}(X_y, E|_{X_y}) < \infty$]

<u>Th.</u> $\mathbb{H}^{p,q}(E) \longrightarrow Y$ is a <u>hol. v.b.</u> with fibre $H^{p,q}(F, E_0)$. Let $A^{p,q}(X,E)$ = space of C^∞ E-valued forms of type (p,q) on X. For $p \geq 0$ fixed $\exists$ a decreasing filtation of $\{A^{p,*}(X,E), \bar\partial\}$, hence $\exists$ a (B-L) s.s. $\{{}^{(p)}E_r^{s,t}\}$. This converges to the $\bar\partial$ cohomology $H^{p,*}(X,E)$, with

$$\boxed{{}^{(p)}E_2^{s,t} = \bigoplus_{i\geq 0} H^{i,s-i}(Y, \mathbb{H}^{p-i,t+i}(E))}$$

$\swarrow$ cohomology bundle

Thus cohomology of X is related to the cohomologies of Y and F

Examples

(i) If $E = \pi^* V$ for $V \xrightarrow{\text{hol. v.b.}} Y$

then $\begin{array}{c} E \longrightarrow X \\ \pi \downarrow \\ Y \end{array}$ is a B-L diagram

$$\left[\begin{array}{ccc} E = \pi^* V & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & Y \end{array}\right]$$

$\boxed{\text{B-L s.s.} = \text{Borel s.s.}}$

$p = 0$ gives Leray s.s.

(ii) Let $Z \xrightarrow{\text{hol. principal } H_1 \text{ bundle}} Y$, $\left[\begin{array}{l} H_1 = \text{complex} \\ \text{Lie group} \end{array}\right]$

$X = Z/H_2$ = orbit space for $H_2 \overset{\text{closed } \mathbb{C} \text{ s.g.}}{\subset} H_1$ [for restricted right action] $\ni F = H_1/H_2$ is compact. Assume $Z \to X$ is a hol. principal H_2 bundle. For a f.d. hol. H_2-module V_0, let $E \overset{\text{def.}}{=} Z \times_{H_2} V_0 \to X$ be the associated hol. v.b. Then

$\begin{array}{c} E \longrightarrow X \\ \pi \downarrow \\ Y \end{array}$ is a B-L diagram $\boxed{\text{with } E_0 = H_1 \times_{H_2} V_0 \longrightarrow H_1/H_2 = F}$

well-defined since $H_2 \subset H_1$: $\begin{array}{c} Z \to X \\ \searrow \downarrow \\ Y \end{array}$ commutes

Also the cohomology bundle <u>can</u> <u>be</u> <u>computed</u>, keeping in mind $\exists$ a <u>rep.</u> of H_1 on the Dolbeault cohomology $H^{p,q}(H_1/H_2, E_0)$ [again $E_0 = H_1 \times_{H_2} V_0$] ;

$$\boxed{\mathbb{H}^{pq}(E) \simeq Z \times_{H_1} H^{p,q}(H_1/H_2, E_0)} \Longrightarrow$$

(p) E_2^{st} recall $= \bigoplus_i H^{i,s-i}(Y, \mathbb{H}^{p-i,t+i}(E))$

$\overset{!!}{\swarrow}$ <u>generalization</u> <u>of</u> <u>R. Bott</u> (case $p=0$)

$$= \bigoplus_{i \geq 0} H^{i,s-i}(Y, Z \times_{H_1} H^{p-i,t+i}(H_1/H_2, E_0))$$

[recall: s.s. $\Longrightarrow H^{p,*}(X, Z \times_{H_2} V_0 \overset{\text{def.}}{=} E)$, $X \overset{\text{def.}}{=} Z/H_2$]

(iii)
$$\begin{array}{ccc} \Gamma\backslash\mathcal{L}_\lambda & \xrightarrow{\text{hol. line bundle}} & \Gamma\backslash G/H \\ & & \downarrow \\ & & \Gamma\backslash G/K \end{array}$$

is a B-L diagram for

G non-cpt. s.s.
$|$
K max. cpt.
$|$
H compact Cartan in G

$\Gamma \subset G$ discrete

$\boxed{\text{More on this later!}}$

$\boxed{\mathcal{L}_\lambda = G \times_H e^\lambda \to G/H,}$
$\boxed{G/K \text{ Hermitian symmetric}}$
$\boxed{\Gamma \text{ acting freely on } G/H}$

Proof of a Langlands' conjecture

[going well beyond it]

G connected, non-compact, semisimple, linear

$|$

K maximal compact

$|$

H compact Cartan in G / i.e. $\hat{G}_d \neq \phi$ Harish-Chandra

$\Gamma \subset G$ discrete, $\Gamma \backslash G$ compact

[later we drop compactness]

L = lattice $\{ d\chi \mid \chi \in \hat{H} \} \supset \Delta = \Delta(g, h) =$ non-zero roots, $\Delta^+ \subset \Delta$, $2\delta = \sum_{\alpha \in \Delta^+} \alpha$

$\boxed{n \overset{\text{def.}}{=} \sum_{\alpha \in \Delta^+} g_{-\alpha}}$ = anti-hol. tangent vectors at $o = 1H$ for

a unique G-invariant hol. structure

on G/H.

For $\lambda \in L$ fixed $\exists$ following data :

(i) $\mathcal{L}_\lambda = G \times_H e^\lambda \longrightarrow G/H$ [hol. line bundle]

(ii) $\Gamma \backslash \mathcal{L}_\lambda \longrightarrow X_\Gamma \overset{\text{def.}}{=} \Gamma \backslash G/H$ [hol. line bundle]

(iii) $H^q(X_\Gamma, \Gamma\backslash\mathcal{L}_\lambda)$ = q^{th} dim. cohomology of X_Γ with coeff. in the <u>sheaf</u> <u>of</u> <u>local</u> <u>hol.</u> <u>sections</u> <u>of</u> $\Gamma \backslash \mathcal{L}_\lambda$

(iv) $q(\lambda) \overset{\text{def.}}{=} |\{\alpha \in \Delta_k^+ \mid (\lambda+\delta, \alpha) < 0\}| + |\{\alpha \in \Delta_n^+ \mid (\lambda+\delta, \alpha) > 0\}|$

(v) Harish-Chandra d.s. rep. $\pi_{\lambda+\delta} \in \hat{G}_d \subset \hat{G}$, for $\lambda+\delta$ <u>regular</u> $\leftarrow$ assume

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R. Langlands' conjecture [LC] :

[for $\Gamma\backslash G$ compact, Γ acting <u>freely</u> on G/H]

(a) For λ "sufficienty non-singular"/? [say, at least, P. Griffiths sense]

$H^q(X_\Gamma, \Gamma\backslash\mathcal{L}\lambda) = 0$ for $q \neq q(\lambda)$

(b) $\dim H^{q(\lambda)}(X_\Gamma, \Gamma\backslash\mathcal{L}\lambda) = m_{\lambda+\delta}(\Gamma) \overset{\text{def.}}{=}$ <u>multiplicity</u> of $\boxed{\text{integrable}}$ $\pi_{\lambda+\delta}$ in $L^2(\Gamma\backslash G)$

Remarks on proof of LC

(i) R. Langlands computes $m_{\lambda+\delta}(\Gamma)$ for hol. d.s. $\pi_{\lambda+\delta}$, via Selberg trace formula/STF

(ii) W. Schmid proves LC in part - for $|(\lambda,\alpha)|>b$ $\forall \alpha\in\Delta$ [for suitable b, after P. Griffiths], but does this $\Rightarrow$ integrability of $\pi_{\lambda+\delta}$?

(iii) F. Williams finds full proof, not only for all integrable d.s., but for ∞-many non-integrable d.s.. Best possible result [as shown by an ASF] but highly dependent of results of W. Schmid / his lowest K-type theorem for example [also due to N. Wallach],

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Proof does not use STF - but spectral sequences! [more on this later]

(ii) is replaced by $(\lambda+\delta-\delta^{(\lambda)},\alpha)>0$ $\forall\alpha\in P_n(\lambda)$ for $P(\lambda)=\{\alpha\in\Delta\,|\,(\lambda+\delta,\alpha)>0\}$

Also best possible vanishing theorem for the cohomology $H^q(X_n, \Gamma\backslash\mathcal{L}_\lambda)$

Beyond LC / G. Warner's 3rd problem

1st question: What replaces $q(\lambda)$?

How much higher in cohomology does one travel? / oxygen mask needed?

$$m_\pi(\Gamma) = \dim H^{q(\lambda)+?}(\Gamma\backslash G/H, \Gamma\backslash \mathcal{L}_\lambda), \ \pi \notin \hat{G}_d$$

For example assume G/K is Hermitiam symmetric - with a G-invariant hol. structure compatible with Δ^+: every $\alpha \in \Delta_n^+$ is totally positive (t.p.): $\alpha+\beta \in \Delta$ for $\beta \in \Delta_K \Rightarrow \alpha+\beta \in \Delta_n^+$ [$\mathfrak{p}^\pm = \sum_{\alpha\in\Delta_n^+} \mathfrak{g}_{\pm\alpha}$ are abelian K-invar.]

For a Parthasarathy unitarizable highest weight rep. $\pi = \pi(q) \in \hat{G}$

$$m_\pi(\Gamma) = \dim H^{q(\lambda)+(|\Delta_n^+|-|q_{u,n}|)}(\ , \ ),$$

where $q_{u,n} \overset{\text{def.}}{=}$ non-compact roots in the unipotent radical of the parabolic $\mathfrak{q}$:

$\boxed{\text{Let } \overline{\Delta^+} \overset{\text{def.}}{=} \Delta^+_k \cup -\Delta^+_n}$

By a <u>Parthasarathy</u> unitarizable $\overline{\Delta^+}$ - highest weight rep. we mean one with a <u>regular</u> ∞ - <u>character</u>

= <u>unique</u> <u>irred.</u> <u>quotient</u> of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(k+\mathfrak{p}^-)} V_\mu$, $V_\mu \in \hat{K}$ with Δ^+_k - highest wgt. μ, with $(\mu + \delta_k - \delta_n, \alpha) \neq 0$ $\forall\, \alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$; $\frac{2(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ $\forall\, \alpha \in \Delta$

$$\boxed{m_\pi(\Gamma) = \dim H^{q(\lambda) + (|\Delta^+_n| - |q_{u,n}|)}\left(\Gamma\backslash G/H, \Gamma\backslash \mathcal{L}_\lambda\right)}$$

if (at least) $(\lambda + \delta - \delta^{(\lambda)}, \alpha) \neq 0$ for $\alpha \in q_{u,n} \cup Q_\lambda$, $Q_\lambda \overset{\text{def.}}{=} \{\alpha \in \Delta^+_n \mid (\lambda + \delta, \alpha) > 0\}$

<u>General</u> classification / construction of unitarizable highest wgt. rep. due to T. Enright - R. Howe - N. Wallach, H. Garland - G. Zuckerman, H. Jakobsen, D. Vogan, M. Vergne, M. Kashiwara, H. Rossi, K. Gross, W. Holman, R. Kunze, etc.

Proof uses that

$$\begin{array}{ccc} \Gamma\backslash \mathcal{L}_\lambda & \longrightarrow & \Gamma\backslash G/H \\ & & \downarrow \\ & & \Gamma\backslash G/K \end{array}$$

is a B-L diagram

[even when $\Gamma\backslash G$ is non-compact]

with a nice formula for its cohomology bundle ∴ a nice formula for the E_2 spectral sequence terms

In fact collapse of spectral sequence by Borel-Weil theorem :

suitable sheaf

$$\boxed{H^q(\Gamma\backslash G/H, \Gamma\backslash \mathcal{L}_\lambda) = H^{q-q_0}(\Gamma\backslash G/K, \mathcal{O}_\lambda)}$$

$\forall q \geq 0$, $q_0 = |\{\alpha \in \Delta^+_K \mid (\lambda + \delta_K, \alpha) < 0\}|$

[this is due to M. Ise for $\Gamma\backslash G$ compact]

To conclude proof apply an ASF (F. Williams) for multiplicities $m_{\pi(q)}(\Gamma)$ of Parathasarathy rep. $\pi(q)$

[for the locally symm. space $\Gamma\backslash G/K$] again no STF

Multiplicity of derived functor modules

In the previous discussion, H-C d.
and Parthasarathy highest wgt. rep.
are special cases of rep. π with

non-trivial $(\mathfrak{g}, K)$ cohomology

$$\boxed{\pi \overset{\text{"}}{=} \pi(\lambda,\mathfrak{q}) = \mathcal{R}_{\mathfrak{q}}^{\dim(\mathfrak{u}\cap\mathfrak{k})} \mathbb{C}_{\lambda+\delta-\delta^{(\lambda)}}}$$

derived functor module / G. Zuckerman, D. Vogan

for a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ with

Cartan involution (θ); Levi ($\mathfrak{l}$); unipot. rad ($\mathfrak{u}$)

$(\lambda+\delta-\delta^{(\lambda)}, \Delta(\mathfrak{l})) = 0$;

Zuckerman coh. parabolic induction

$$\mathcal{R}_{\mathfrak{q}}^{*} : \mathrm{Cat}(\mathfrak{l}, L\cap K) \longrightarrow \mathrm{Cat}(\mathfrak{g}, K)$$

Problem : Find a cohomological meaning

of $m_{\pi(\lambda,\mathfrak{q})}(\Gamma)$, [now more generally] $\mathrm{vol}(\Gamma\backslash G) < \infty$

in the discrete spectrum $L^2_d(\Gamma\backslash G)$:

$$\boxed{m_{\pi(\lambda,\mathfrak{q})}(\Gamma) = \dim H_2^{n(\pi)=?}(\Gamma\backslash G/H, \Gamma\backslash \mathcal{L}_\lambda) \ ?}$$

L_2 - cohomology replaces sheaf coh.

Sketch of solution / F.W.

Recall $\boxed{n \overset{\text{def.}}{=} \sum_{\alpha \in \Delta^+} g_{-\alpha}}$ [= space of anti-hol. tangent vectors at $o \in G/H$]

Using strong results of W. Schmid (also W. Casselman, S. Osborne)

$$\boxed{\begin{array}{l} H_2^r(\Gamma\backslash G/H, \Gamma\backslash \mathcal{L}_\lambda) = \sum_{\substack{\pi \in \hat{G},\ \pi(\Omega) = (\lambda, \lambda+2\delta)1}} m_\pi(\Gamma) H^r(n, H_\pi)_{e^{-\lambda}}, \quad \forall r \geq 0 \\ \text{a type of } G/H \text{ Matsushima formula} \end{array}}$$

K-finite vectors (H_π); Casimir (Ω)

∴ for $\lambda \in L$ with $\lambda+\delta$ regular, find all $\pi \in \hat{G}$ with $\pi(\Omega) = (\lambda + \lambda+2\delta)1$ and $r \ni H^r(n, H_\pi)_{e^{-\lambda}} \neq 0$. Here [as in dis. case] an additional mild condition on λ

Answer:

$\Longleftrightarrow$ (i) π = derived functor module

(ii) $r = r(\pi)$ = an explicit integer determined by π [generalization of Langlands' $\mathcal{E}(\lambda)$], in which case $\dim H^{r(\pi)}(n, H_\pi)_{e^{-\lambda}} = 1$

$\boxed{\text{Why}}$?

Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ $\left[\begin{array}{l}\text{Cartan}\\ \text{decom.}\end{array}\right]$, $\mathfrak{p}^- \overset{\text{def.}}{=} \sum_{\alpha \in \Delta_n^+} \mathfrak{g}_{-\alpha}$,

$H_\pi = \sum_{\tau \in \hat{K}} \oplus\, m_\tau V_\tau$ $\left[\begin{array}{l}\text{primary}\\ K\text{-decom.}\end{array}\right]$

Study 2 spectral sequences :

$\boxed{1.}$ ${}^{(\pi)}E \Longrightarrow H^*(\mathfrak{n}, H_\pi)_{e^{-\lambda}}$ $\left[\text{for any } \pi \in \hat{G}\right]$

$\boxed{\text{Hochschild-Serre for } \mathfrak{k}\cap\mathfrak{n} \subset \mathfrak{n}}$

with ${}^{(\pi)}E_1^{rs} = H^s(\mathfrak{k}\cap\mathfrak{n}, H_\pi \otimes \Lambda^r \mathfrak{p}/\mathfrak{p}^-)_{e^{-\lambda}}$

$\therefore\; = \sum_{\tau\in\hat{K}} \oplus\, m_\tau H^s(\mathfrak{k}\cap\mathfrak{n}, V_\tau \otimes \Lambda^r \mathfrak{p}/\mathfrak{p}^-)_{e^{-\lambda}}$

$\boxed{2.}$ ${}^{(r)}E \Longrightarrow H^*(\mathfrak{k}\cap\mathfrak{n}, V_\tau \otimes \Lambda^r \mathfrak{p}/\mathfrak{p}^-)_{e^{-\lambda}}$

$\boxed{\text{a finite filtration of } \Lambda^r \mathfrak{p}/\mathfrak{p}^-}$

with ${}^{(r)}E_1^{ab} = H^{a+b}(\mathfrak{k}\cap\mathfrak{n}, V_\tau)_{-\lambda-\sigma_{N-a}}$ for

wgts. $\{\sigma_i\}_{i=1}^{N=\binom{n}{r}} = \{\langle T\rangle \mid T \subset \Delta_n^+,\ |T| = r\}$, $n = |\Delta_n^+|$

Here $H^*(\mathfrak{k}\cap\mathfrak{n}, V_\tau)$ is computed by B. Kostant's famous "$\mathfrak{n}$"-coh. theorem

: $\mathfrak{b}_K \overset{\text{def.}}{=} \mathfrak{h} \oplus (\mathfrak{k}\cap\mathfrak{n})$ = Borel sub. alg. of $\mathfrak{k}$.

<u>In</u> <u>the</u> <u>end</u> : <u>same</u> type formula as for hol. highest wgt. modules $\left[\begin{array}{l}\text{with regular} \\ \infty\text{-character}\end{array}\right]$:

$$\boxed{\begin{array}{l} \text{For } \quad \pi = \pi(\lambda, \mathfrak{q}) = \mathcal{R}_{\mathfrak{q}}^{\dim(\mathfrak{u}\cap\mathfrak{k})}\left(\Gamma_{\lambda+\delta-\delta(\lambda)}\right) \in \hat{G} \\[2em] m_\pi(\Gamma) = \dim H_2^{\,\mathfrak{q}(\lambda)+|\Delta_n^+|-|\mathfrak{q}_{u,n}|}\left(\Gamma\backslash G/H, \Gamma\backslash \mathcal{L}_\lambda\right) \end{array}}$$

under mild generic conditions on λ, similar to hol. case.

Remarks

(i) By product of preceding arguments gives $\boxed{n\text{-coh.}}$ of discrete series rep. [W. Schmid] and limits of discrete series [F. W.] - by a more careful analysis of spectral seq. / since $H^*(n, H_\pi) = \sum_{\mu \in L} H^*(n, H_\pi)_{e^{-\mu}}$

(ii) also have coh. meaning of mul. of holomorphic limits of d.s.

(iii) Apart from d.s. case, these results do not compute $L^2(\Gamma\backslash G)$ multiplicities - only provide a cohomological meaning. Have survival of coh. in "other" dimensions ∴ Atiyah-Singer formulas do not settle matter

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(iv) In the full proof (and beyond) of LC we also make key use of P. Trombi-V. Varadarajan estimate for integrable d.s. $\pi_{\lambda+\delta}$: $2|(\lambda+\delta, \alpha)| > \sum_{\beta \in P(\lambda)} |(\alpha, \beta)|$, $\alpha \in P_n^{(\lambda)}$.

BGG type resolution / applications

Assume (again) G/K has a G-invar. hol. structure corres. to Δ^+:

every $\alpha \in \Delta_n^+$ is t.p. $\Leftrightarrow$ $\mathfrak{p}^{\pm} \overset{\text{def.}}{=} \sum_{\alpha \in \Delta_n} \mathfrak{g}_{\pm\alpha}$

= spaces of hol./anti hol (+/−) tangent vec. at $o \in G/K$ [= K-modules]

Again let $\mathfrak{q} = \underset{\text{Levi}}{\mathfrak{l}} + \underset{\text{unipot. rad.}}{\mathfrak{u}} \supset \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \overset{\text{def.}}{=} \mathfrak{b}_{\Delta^+}$

be a θ-stable parabolic,

L = corres. (connected) Levi s.g. of G

$W(\mathfrak{l}) \overset{\text{def.}}{=}$ s.g. of $W(\mathfrak{g}, \mathfrak{h})$ gen. by $\Delta(\mathfrak{l})$-reflec

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$W^1(\mathfrak{l}) \overset{\text{def.}}{=} \{ w \in W(\mathfrak{l}) \mid \Delta_K^+ \cap \Delta(\mathfrak{l}) \subset w(\Delta^+ \cap \Delta(\mathfrak{l})) \}$

$\ell(w) \overset{\text{def.}}{=} | \{ \alpha \in \Delta^+ \cap \Delta(\mathfrak{l}) \mid w^{-1}\alpha \in -(\Delta^+ \cap \Delta(\mathfrak{l})) \} |$

$\mathfrak{l}$ - length of $w \in W(\mathfrak{l})$.

Given a f.d. irreducible L module F_L, form the <u>produced</u> / <u>induced</u> $(\mathfrak{g}, B)$ modules for $B \overset{\text{def.}}{=} L \cap K$:

$$\boxed{\begin{aligned} \mathrm{Pro}^{\mathfrak{g},B}_{\bar{\mathfrak{q}},B} F_L &\overset{\text{def.}}{=} \mathrm{Hom}_{U(\bar{\mathfrak{q}})}(U(\mathfrak{g}), F_L)_{B\text{-finite}} \\ \mathrm{Ind}^{\mathfrak{g},B}_{\mathfrak{q},B} F_L &\overset{\text{def.}}{=} U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F_L \end{aligned}}$$

Here $\bar{u} \cdot F_L = 0$, $u \cdot F_L = 0$

<u>Th</u>$_1$ (F. W$_1$) $\exists$ a $(\mathfrak{g}, B)$ <u>resolution</u> of $\mathrm{Pro}^{\mathfrak{g},B}_{\bar{\mathfrak{q}},B} F_L$ of BGG type :

(exact sequence)

$$\boxed{0 \to \mathrm{Pro}^{\mathfrak{g},B}_{\bar{\mathfrak{q}},B} F_L \to V_0 \to V_1 \to \cdots \to V_{\dim(\ell \cap p^+)} \to 0}$$

where $V_{j \geq 0} = \sum\limits_{\substack{w \in W^1(\ell) \\ \ell(w) = j}} \oplus \; \mathrm{Pro}^{\mathfrak{g},B}_{\overline{\mathfrak{q} \cap (k + p^+)},B} F_B(w)$

$\boxed{F_B(w) \in \hat{B}}$

Also there exists, similarly, a BGG resolution of $\mathrm{Ind}^{\mathfrak{g},B}_{\mathfrak{q},B} F_L$.

(a) Theorem is due to $\boxed{\text{R. Stanke}}$ when
$G = SU(p,q)$, $K = S(U(p) \times U(q))$

(b) Proof uses a generalized version of BGG for $\mathfrak{l}^{ss} \overset{\text{def.}}{=} [\mathfrak{l},\mathfrak{l}]$ $\boxed{\text{due to J. Lepowsky}}$

(c) $\exists$ a $\boxed{\text{T. Enright - N. Wallach}}$ resolution of Verma (= induced) modules [in $Cat(\mathfrak{g})$ only]
$\neq$ above resolution

Given above BGG resolution of $\operatorname{Pro} F_L$ $\in Cat(\mathfrak{g}, B)$ $\exists$ a Grothendieck 1st quadrant s. s. $[E_2^{rs} = 0$ if r or $s < 0]$ for

$\Gamma^{\mathfrak{g},K}_{\mathfrak{g},B=L\cap K} : Cat(\mathfrak{g}, B) \to Cat(\mathfrak{g}, K)$ $\boxed{\text{Zuckerman functor}}$

: $E^{**}_{*} \Longrightarrow \Gamma^{*} \operatorname{Pro} F_L$ [higher derived functors]

$\boxed{\text{since } Cat(\mathfrak{g}, B) \text{ has "enough injectives}}$ $\Longleftarrow$

See, for example, a more general version of Grothendieck in Appendix D of A. Knapp - D. Vogan Princeton text.

Computation of Grothendieck s.s. $\boxed{E_1^{rj}}$

main tools $\boxed{\text{chap. 6 / D. Vogan's green monster}}$:

(i) algebraic induction in stages :

$$\mathrm{Pro}^{\mathfrak g,\, B \overset{\text{def.}}{=} L\cap K}_{\overline{\mathfrak q\cap(\mathfrak k+\mathfrak p^+)},\, B} F = \mathrm{Pro}^{\mathfrak g, B}_{\mathfrak k+\mathfrak p^-,\, B}\left(\mathrm{Pro}^{\mathfrak k, B}_{\overline{\mathfrak q\cap\mathfrak k},\, B}\, \mathrm{Res}^{\overline{\mathfrak q\cap\mathfrak k},\, B}_{\overline{\mathfrak q\cap(\mathfrak k+\mathfrak p^+)},\, B} F\right)$$

$F=(\mathfrak l\cap\mathfrak k, B)$ module	$\overline{\mathfrak q\cap\mathfrak k} \subset \overline{\mathfrak q\cap(\mathfrak k+\mathfrak p^+)} = (\mathfrak l\cap\mathfrak k)\oplus(\bar{\mathfrak u}+\mathfrak p^-)$

(ii) Γ^j, Pro commutation : $\left(\Gamma^{\mathfrak g, K}_{\mathfrak g, B}\right)^j \mathrm{Pro}^{\mathfrak g, B}_{\overline{\mathfrak q\cap(\mathfrak k+\mathfrak p^+)},\, B} F$

$$= \mathrm{Pro}^{\mathfrak g, K}_{\mathfrak k+\mathfrak p^-,\, B} \left(\Gamma^{\mathfrak k, K}_{\mathfrak k, B}\right)^j \left(\mathrm{Pro}^{\mathfrak k, B}_{\overline{\mathfrak q\cap\mathfrak k},\, B}\, \mathrm{Res}^{\overline{\mathfrak q\cap\mathfrak k},\, B}_{\overline{\mathfrak q\cap(\mathfrak k+\mathfrak p^+)},\, B} F\right)$$

(iii) algebraic Borel-Weil $\boxed{\text{T. Enright and N. Wallach}}$

: computes $\left(\Gamma^{\mathfrak k, K}_{\mathfrak k, B}\right)^j \left(\mathrm{Pro}^{\mathfrak k, B}_{\overline{\mathfrak q\cap\mathfrak k},\, B} F_1\right)$ $\boxed{\text{example: } F_1 = \mathrm{Res}^{\overline{\mathfrak q\cap\mathfrak k},\, B}_{\overline{\mathfrak q\cap(\mathfrak k+\mathfrak p^+)},\, B} F}$

(iv) Γ^j, $\Sigma\oplus$ commutation , where

$$\boxed{E_1^{rj} = \left(\Gamma^{\mathfrak g, K}_{\mathfrak g, B}\right)^j V_r \quad r,s\geq 0} \doteq \sum_{\substack{w\in W^1(\mathfrak l)\\ \ell(w)=r}}\oplus \left(\Gamma^{\mathfrak g, K}_{\mathfrak g, B}\right)^j \mathrm{Pro}^{\mathfrak g, B}_{\overline{\mathfrak q\cap(\mathfrak k+\mathfrak p^+)},\, B} F_B(w)$$

Grothendieck $\boxed{\text{by def. of } V_r}$

$\therefore$ by (ii), (iii) $\left(\Gamma^{\mathfrak g, K}_{\mathfrak g, B}\right)^j \mathrm{Pro}^{\mathfrak g, B}_{\overline{\mathfrak q\cap(\mathfrak k+\mathfrak p^+)}} F_B(w)$ is computed by Borel-Weil

In the end:

$$E_1^{ab} = \sum \oplus \, \mathrm{Pro}_{\mathfrak{k}+\mathfrak{p}^-, K}^{\mathfrak{g}, K} \; F_K(\sigma_w w(\lambda+\delta)-\delta)$$

$w \in W^1(\mathfrak{l})$, $\ell(w) = a$, $w(\lambda+\delta)$ is $\Delta_{\mathfrak{k}}$-regular

unique $\sigma_w \in W_{\mathfrak{k}} = W(\mathfrak{k}, \mathfrak{h}) \ni (\sigma_w w(\lambda+\delta), \Delta_{\mathfrak{k}}^+) > 0$

$\ell(\sigma_w) = b$

---

where $F_K(\mu) \in \hat{K}$ has $\Delta_{\mathfrak{k}}^+$-highest wgt. $\mu \in \mathfrak{h}^*$

$\lambda \in \mathfrak{h}^*$ is the $\Delta^+ \cap \Delta(\mathfrak{l})$-highest wgt. of a

f.d. irreducible $(\mathfrak{l}, L)$ module $F_L(\lambda)$.

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one can show

Cor. [Vanishing theorem / preceding notation]

If $\left(\Gamma_{\mathfrak{g}, L\cap K}^{\mathfrak{g}, K}\right)^j \mathrm{Pro}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K} F_L(\lambda) \neq 0$

then $\exists \, (w, \sigma_w) \in W^1(\mathfrak{l}) \times W_{\mathfrak{k}}$ above $\ni$

$$j = a+b = \ell(w) + \ell(\sigma_w) \overset{\text{Prop.}}{=} \ell(\sigma_w w) \overset{\text{def.}}{=}$$

$$\left| \{ \alpha \in \Delta^+ \mid (\sigma_w w)^{-1}\alpha \in -\Delta^+ \} \right| \leq \dim(\mathfrak{u} \cap \mathfrak{k}),$$

$\rightarrow$ D. Vogan – R. Blattener, page 363 of green monster

In particular $\left[\begin{array}{c}\text{classical} \\ \text{Vogan-Zuckerman} \\ \text{vanishing}\end{array}\right]$:

$\boxed{\text{collapse of spectral sequence}}$

If $(\lambda+\delta, \alpha) < 0 \quad \forall \alpha \in \Delta(u)$, then

$$\left(\Gamma^{g,K}_{g,B=L\cap K}\right)^j \mathrm{Pro}^{g,B}_{\bar{q},B} F_L(\lambda) = 0 \quad \text{for}$$

$j \neq s \overset{\text{def.}}{=} \dim(u \cap k)$ $\left[\begin{array}{c}\text{the middle} \\ \text{dimension}\end{array}\right]$;

$\boxed{s \text{ used in } L^2(\Gamma\backslash G) \text{ mul.}}$

Program / Problem :

Construct irreducible unitarizable (g, K) modules $\left(\Gamma^{g,K}_{g,B=L\cap K}\right)^j \mathrm{Pro}^{g,B}_{\bar{q},B} F_L(\lambda)$

with j below the middle dimension

$\boxed{j < s}$; thus $(\lambda+\delta, \alpha) \geq 0$ for some $\alpha \in \Delta(u)$; say $\lambda =$ a reduction point in the Enright-Howe-Wallach classification of unitary highest weight modules.