Orbit graphs of associated varieties

—joint work with

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Plan of talk

1. Motivation & Problems
2. Orbit graph
   Introduce orbit graph of nilpotent orbits for a symmetric pair (G, K)
3. Description of orbit graphs
   Give a combinatorial description of orbit graphs
4. Induction of orbit graphs
   Define induction of orbit graphs, related to connectedness in codimension one
5. Associated varieties of HC-modules
   Show connected components of orbit graphs are associated varieties of certain HC-modules

Nilpotent orbits

G : reductive algebraic group / $\mathbb{C}$

$K \subset G$: symmetric subgrp $\leftrightarrow$ involution θ

Example of Symmetric Pair (G, K)

$G = U(p,q) = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^t\overline{g} I_{p,q} g = I_{p,q}\}$ $\quad I_{p,q} = \mathrm{diag}(1_p, -1_q)$

$$K = U(p) \times U(q) \xhookrightarrow{\text{diag embedded}} G$$

$\mathfrak{g} := \mathrm{Lie}\,(G)_{\mathbb{C}} \rightsquigarrow \quad \underset{\theta}{\mathfrak{g}} = \underset{(+1)}{\mathfrak{k}} \oplus \underset{(-1)}{\mathfrak{s}}$: Cartan decomp

Nilpotent variety : $\mathcal{N}(\mathfrak{s}) = \mathfrak{s} \cap \mathcal{N}(\mathfrak{g})$ $\qquad G \curvearrowright \mathcal{N}(\mathfrak{g}), \ \ K \curvearrowright \mathcal{N}(\mathfrak{s})$

Fact 1 : nilpotent variety

- $\#\mathcal{N}(\mathfrak{g})/\operatorname{Ad} G < \infty$: $\exists$ fin many $\#$ of G-orbits
- $\#\mathcal{N}(\mathfrak{s})/\operatorname{Ad} K < \infty$: $\exists$ fin many $\#$ of K-orbits

$\mathcal{O} \in \mathcal{N}(\mathfrak{g})/G$: nilpotent G-orbit $\rightsquigarrow$
$\mathcal{O} \cap \mathfrak{s} = \mathbb{O}_1 \sqcup \cdots \sqcup \mathbb{O}_\ell$: decomp into conn comp

Fact 2 : nilpotent orbits for symmetric pair

- $\forall \mathbb{O}_i$ is K-orbit
- $\mathcal{O}$ is a symplectic G-variety (Kirillov-Kostant symplectic form)
- $\mathbb{O}_i \subset \mathcal{O}$ is Lagrangian subvariety, $\dim \mathbb{O}_i = \frac{1}{2} \dim \mathcal{O}$ in particular

(π, X) : irreducible HC $(\mathfrak{g}, K)$-module
$\rightsquigarrow$ $\mathcal{AV}(X)$: associated variety $\subset \mathcal{N}(\mathfrak{s})$, K-stable

Fact 3: Associated Variety

- $\mathcal{AV}(\mathrm{Ann}\, X) = \overline{\mathcal{O}}$: irreducible
- $\mathcal{AV}(X) = \cup_{j=1}^{r} \overline{\mathbb{O}}_{i_j}$: irreducible decomp (reducible in general)

Vogan's theorem & Definition of orbit graph

Notation: $\partial \mathbb{O} = \overline{\mathbb{O}} \setminus \mathbb{O}$ codim $\partial \mathbb{O}$: codim of $\partial \mathbb{O}$ in $\overline{\mathbb{O}}$

Theorem (Vogan, 1991)

Suppose
- (π, X) : *irreducible HC* $(\mathfrak{g}, K)$*-module*
- $\mathcal{AV}(X)$ *is reducible*

$\Longrightarrow$ codim $\partial \mathbb{O}_i = 1$ *for* $\forall \overline{\mathbb{O}}_i \subset \mathcal{AV}(X)$: *irreducible component*

$\therefore$ We are interested in the boundaries of $\{\overline{\mathbb{O}}_i\}_i$ which are codimension one $\rightsquigarrow$ Orbit Graph $\Gamma_K(\mathcal{O})$

$\mathcal{O} \in \mathcal{N}(\mathfrak{g})$ $\mathcal{O} \cap \mathfrak{s} = \mathbb{O}_1 \sqcup \cdots \sqcup \mathbb{O}_\ell$: decomp into K-orbits

Definition ($\Gamma_K(\mathcal{O})$: orbit graph)

- Vertices : $\mathscr{V} = \{\mathbb{O}_i \mid 1 \leq i \leq \ell\}$: nilpotent K-orbits
- Edges : $\mathbb{O}_i - \mathbb{O}_j \iff$ codim $\partial \mathbb{O}_i \cap \partial \mathbb{O}_j = 1$ in $\overline{\mathbb{O}}_i$ (or $\overline{\mathbb{O}}_j$)

Structure of Orbit Graph

■ How to find out the tructure of $\Gamma_K(\mathcal{O})$?

Strategy:

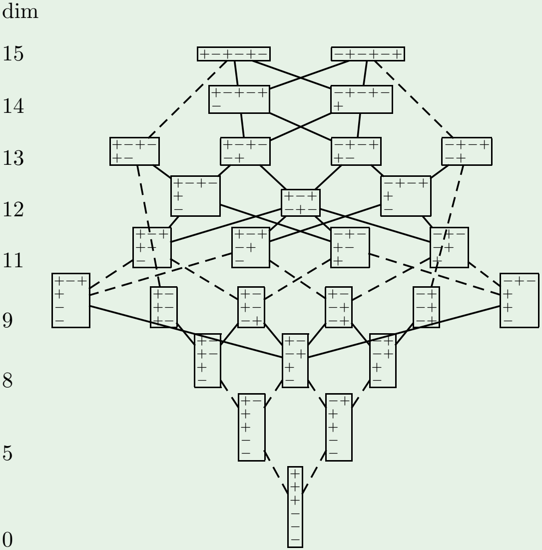
1. Use Ohta's description of closure ordering
   [T. Ohta, Tohoku Math J, 1991]
2. Calculate dimension and determine codim-one-adjacency
   $\rightsquigarrow$ Find a combinatorial description
3. Embedding of $\Gamma_K(\mathcal{O})$ into Euclidean space $\mathbb{R}^N$
4. Induction of graphs from low rank case

Skip (1) & (2) $\rightsquigarrow$ want to describe (3) & (4) in this talk

type AIII

We are concentrating on type AIII case:
$G = \mathrm{GL}_n \supset K = \mathrm{GL}_p \times \mathrm{GL}_q \ \ (n = p + q) \longleftrightarrow G_{\mathbb{R}} = \mathrm{U}(p, q)$

Closure ordering of nilpotent K-orbits for $(GL_6, GL_3 \times GL_3)$



Naïve but Natural Questions

1. How many vertices in $\Gamma_K(\mathcal{O})$? (i.e., how many K-orbits are contained in a single G-orbit?)
2. Describe explicit structure of $\Gamma_K(\mathcal{O})$
3. When is $\Gamma_K(\mathcal{O})$ connected?
4. Find $\#$ of conn components of $\Gamma_K(\mathcal{O})$
5. Pick $\forall Z \subset \Gamma_K(\mathcal{O})$: conn component
   $\exists$ irreducible HC-module X s.t. $Z = \mathcal{AV}^{\Gamma}(X)$?
6. More generally, pick $Z' \subset \Gamma_K(\mathcal{O})$: connected subgraph
   $\exists$ irreducible HC-module X s.t. $Z' = \mathcal{AV}^{\Gamma}(X)$?

Some remarks:

- Notion of admissible orbits . . . Ohta(1991), Noël(2001)
- Orbits with different shapes can share codim 1 boundary
- Special piece?

Orbit parametrization . . . Signed Young diagram

Young diagram (or partition) $\longleftrightarrow$ nilpotent G-orbits

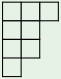
$$\mathrm{YD}(n) = \{\lambda \mid \lambda \vdash n\} \qquad \{\mathcal{O} = \mathcal{O}_\lambda \mid \lambda \in \mathrm{YD}(n)\}$$

signed Young diagram (= SYD) $\longleftrightarrow$ nilpotent K-orbits

$$\mathrm{SYD}(\lambda; p, q) \qquad \{\mathbb{O}_T \mid T \in \mathrm{SYD}(\lambda; p, q)\}$$

Vertices : $\mathscr{V}(\Gamma_K(\mathcal{O}_\lambda)) = \{\mathbb{O}_T \mid T \in \mathrm{SYD}(\lambda; p, q)\}$

Examples of signed Young diagram $(p, q) = (4, 4)$

shape : $\lambda = (3, 2, 2, 1) =$  (Jordan blocks)

SYD :

$+-+$	$+-+$	$+-+$	$-+-$	$-+-$	$-+-$
$+-$	$+-$	$-+$	$+-$	$+-$	$-+$
$+-$	$-+$	$-+$	$+-$	$-+$	$-+$
$-$	$-$	$-$	$+$	$+$	$+$

Express $\lambda \in \mathrm{YD}(n)$ in different manner

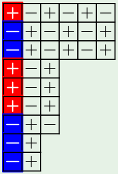
$$\lambda = (i_1, \dots, i_1, i_2, \dots, i_2, \dots, i_k, \dots, i_k) \qquad i_1 > i_2 > \cdots > i_k > 0$$
$$= (i_1^{m(i_1)}, i_2^{m(i_2)}, \dots, i_k^{m(i_k)}) \qquad m(i_j) > 0 \ \ (\text{multiplicity})$$

Among those $m(i)$ rows of length i,

$$m^+(i) = m^+_T(i) \text{ rows begin with } \boxed{+}$$
$$m^-(i) = m^-_T(i) \text{ rows begin with } \boxed{-}$$

so that $m(i) = m^+(i) + m^-(i)$: multiplicity of i

Example $\lambda = (6^3, 3^4, 2^2)$, $(p, q) = (18, 16)$



$$\longleftrightarrow \quad (m^+(6), m^+(3), m^+(2)) = (1, 3, 0)$$

Embedding

Define $\pi : \mathscr{V}(\Gamma_K(\mathcal{O}_\lambda)) \simeq \mathrm{SYD}(\lambda; p, q) \to \mathbb{R}^k$ by

$$\pi(T) = (m^+(i_1), m^+(i_2), \ldots, m^+(i_k)) \in \mathbb{Z}^k_{\geq 0} \subset \mathbb{R}^k$$

where k is # of different parts in λ

$m^+(i_r)$'s must satisfy 2 conditions

1. $0 \leq m^+(i_r) \leq m(i_r) \qquad (1 \leq r \leq k)$,
2. parity condition :
$$p - q = \sum_{i_r \text{ odd}} (m^+(i_r) - m^-(i_r)) = 2 \sum_{i_r \text{ odd}} m^+(i_r) - \sum_{i_r \text{ odd}} m(i_r).$$

Remark

Difference $m^+(i_r) - m^-(i_r)$ contributes only when row length i_r is odd (if even, the same number of $+$'s and $-$'s appear)

Recall the map:

$$\pi : \mathscr{V}(\Gamma_K(\mathcal{O}_\lambda)) \simeq \mathrm{SYD}(\lambda; p, q) \to \mathbb{R}^k$$

$$\pi(T) = (m^+(i_1), m^+(i_2), \ldots, m^+(i_k)) \in \mathbb{Z}^k_{\geq 0}$$

Theorem

The map π so defined is a bijection between

- *nilpotent K-orbits* $\{\mathbb{O}_T \mid T \in \mathrm{SYD}(\lambda; p, q)\}$ *and*
- *$\mathbb{Z}$-lattice points* $(x_i)_{1 \leq i \leq k}$ *in the hyper cube*

$$[0, m(i_1)] \times [0, m(i_2)] \times \cdots \times [0, m(i_k)]$$

satisfying the parity condition :

$$2 \sum_{i_r \text{ odd}} x_r = p - q + \sum_{i_r \text{ odd}} m(i_r)$$

Thus we are left to determine the edges of the orbit graph.

Theorem (Description of Edge)

Two vertices $\mathbb{O}_T, \mathbb{O}_{T'} \in \mathscr{V}(\Gamma_K(\mathcal{O}_\lambda))$ *are connected by edge*

$$\iff \quad \pi(T) - \pi(T') \in \{\pm(e_r - e_{r+1}) \mid 1 \leq r \leq k-1\} \cup \{\pm e_k\}$$

Here $\{e_r = (0, \ldots, 1, \ldots, 0) \mid 1 \leq r \leq k\}$ *denotes the std basis*

Orbit graph for $(p, q) = (6, 6), \quad \lambda = (4, 3, 3, 1, 1)$

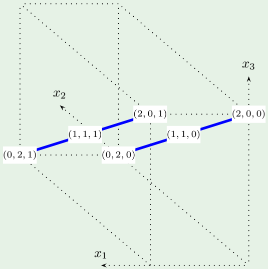
$\mathbb{Z}$-lattice pts (x_1, x_2, x_3) in

$$[0, 1] \times [0, 2] \times [0, 2]$$

parity cond:

$$2(x_2 + x_3) = p - q + \#(\text{odd rows}) = 6 - 6 + 2 + 2 = 4$$

$\rightsquigarrow$ 2-conn components



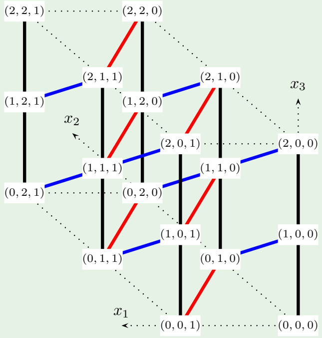
Orbit graph for $(p, q) = (9, 9), \quad \lambda = (6, 4, 4, 2, 2)$

$\mathcal{O}_\lambda$: even nilpotent orbit

$\mathbb{Z}$-lattice pts (x_1, x_2, x_3) in
$[0, 1] \times [0, 2] \times [0, 2]$

parity cond: none
No odd parts!

$\rightsquigarrow$ $\Gamma_K(\mathcal{O}_\lambda)$ is connected



This always happens for even partitions...
$\rightsquigarrow$ $\exists$ generalization

Definition

$\mathcal{O}_\lambda$ is called even if $\forall \lambda_i$'s are all even; or
$\forall \lambda_i$'s are all odd

[More intrinsic definition] $\cdots$ Choose $\{e, h, f\}$: $\mathfrak{sl}_2$-triple corr to $\mathcal{O}$

$$\mathcal{O} \text{ is even} \overset{\text{def}}{\Longleftrightarrow} \text{eigenvalues of } \operatorname{ad} h \text{ are all even}$$

Theorem (even nilpotent orbit)

Consider the symmetric pair $(G, K) = (GL_n, GL_p \times GL_q)$ $(n = p + q)$

1. *If* $\mathcal{O}_\lambda$ *is even, orbit graph* $\Gamma_K(\mathcal{O}_\lambda)$ *is connected*
2. *If* $\exists r$ *s.t.* λ_i : *odd* $(i < r)$, λ_j : *even* $(r \leq j)$ *, then* $\Gamma_K(\mathcal{O}_\lambda)$ *is connected*
3. *These two cases exhaust* $\mathcal{O}_\lambda$ *with connected orbit graph*

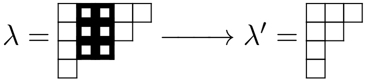
Problem

How to control connected components?

Graph reduction

Given a Young diagram $\lambda \vdash n$

$\rightsquigarrow$ remove two successive columns of the same length from λ to get λ'



h denotes the height of the columns removed

Lemma

$(G, K) = (GL_n, GL_p \times GL_q)$

$(G', K') = (GL_{n-2h}, GL_{p'} \times GL_{q'})$ *with* $(p', q') = (p - h, q - h)$

$\Longrightarrow$ *# of conn compnts of* $\Gamma_K(\mathcal{O}_\lambda)$ = *# of conn compnts of* $\Gamma_{K'}(\mathcal{O}'_{\lambda'})$

This also works for signed Young diagram $T \in \mathrm{SYD}(\lambda; p, q)$

$\rightsquigarrow$ get a reduction map

$$\Phi : \mathrm{SYD}(\lambda; p, q) \to \mathrm{SYD}(\lambda'; p - h, q - h)$$

Graph Induction

$Z' \subset \mathrm{SYD}(\lambda'; p', q')$: subgraph of $\Gamma_{K'}(\mathcal{O}')$
(Identify set of vertices with full subgraph)

Definition

Graph induction of Z' is defined to be

$$\text{g-ind}_{(G',K')}^{(G,K)}(Z') = \text{g-ind}(Z') = \bigcup_{T' \in Z'} \Phi^{-1}(T')$$

Example

$(p,q) = (8,7)$, $(p',q') = (3,2)$; $\lambda' = (2^2, 1) \subset \lambda = (4^2, 3, 2^2)$

$$T' = \begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline + & \\ \cline{1-1}\end{array} \in \mathrm{SYD}(\lambda'; 3, 2) \rightsquigarrow \text{g-ind}(T') \subset \mathrm{SYD}(\lambda; 8, 7)$$

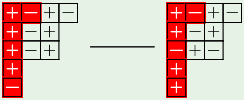
$$\text{g-ind}(T') = \begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline - & + & - & + \\ \hline + & - & + & \\ \hline + & - & & \\ \hline + & - & & \\ \hline\end{array} \text{ —— } \begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline - & + & - & + \\ \hline + & - & + & \\ \hline + & - & & \\ \hline - & + & & \\ \hline\end{array} \text{ —— } \begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline - & + & - & + \\ \hline + & - & + & \\ \hline - & + & & \\ \hline - & + & & \\ \hline\end{array}$$

Example

$(p, q) = (7, 5)$, $(p', q') = (4, 2)$; $\quad \lambda' = (2, 1^4) \subset \lambda = (4, 3^2, 1^2)$

$$T' = \begin{array}{|c|c|} \hline + & - \\ \hline + & \\ \cline{1-1} + & \\ \cline{1-1} + & \\ \cline{1-1} - & \\ \cline{1-1} \end{array} = \begin{array}{|c|c|} \hline + & - \\ \hline + & \\ \cline{1-1} - & \\ \cline{1-1} + & \\ \cline{1-1} + & \\ \cline{1-1} \end{array} \in \mathrm{SYD}(\lambda'; 4, 2) \rightsquigarrow \text{g-ind}(T') \subset \mathrm{SYD}(\lambda; 7, 5)$$

g-ind(T') =



Theorem

g-ind *establishes a bijection between connected components*

$$Z' \subset \Gamma_{K'}(\mathcal{O}'_{\lambda'}) \text{ : conn compnt}$$
$$\Longrightarrow \text{g-ind}(Z') \subset \Gamma_K(\mathcal{O}_\lambda) \text{ : conn compnt}$$

Associated Graph

X : irred HC $(\mathfrak{g}, K)$-module

$\mathcal{AV}(X) = \cup_{i=1}^{r} \overline{\mathbb{O}}_i$: associated variety

$\mathcal{AV}(\mathrm{Ann}(X)) = \overline{\mathcal{O}}_\lambda$: G-orbit corr to X

Definition

Associated graph of X denoted as $\mathcal{AV}^\Gamma(X)$

$\cdots$ full subgraph in $\Gamma_K(\mathcal{O}_\lambda)$ with vertices $\{\mathbb{O}_i \mid 1 \leq i \leq r\}$
those contribute to $\mathcal{AV}(X)$ as irred compnts

Natural question:

- What kind of subgraph is $\mathcal{AV}^\Gamma(X)$?
- Can it be any subgraph?

Almost Theorem

If X is an irreducible HC-module,
associated graph $\mathcal{AV}^\Gamma(X)$ is a connected subgraph of $\Gamma_K(\mathcal{O}_\lambda)$

Theorem (even nilpotent orbit)

$\mathcal{O} \subset \mathfrak{g}$: *even* *nilpotent G-orbit*

1. *Orbit graph* $\Gamma_K(\mathcal{O})$ *is connected*
2. $\exists X$: *irreducible HC-module s.t.* $\Gamma_K(\mathcal{O}) = \mathcal{AV}^{\Gamma}(X)$

   $$\therefore \quad \mathcal{AV}(X) = \overline{\mathcal{O} \cap \mathfrak{s}}$$

   In fact, we can take $X =$ (*deg principal series*) *corr to parabolic subgrp whose Richardson orbit is* $\mathcal{O}$

Example $G_{\mathbb{R}} = U(n, n)$

$P_{\mathbb{R}} = \mathrm{GL}_n(\mathbb{C}) \ltimes N_{\mathbb{R}}$ $\pi_\nu = \mathrm{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}} |\det|^{\nu}$: deg principal series

$$\rightsquigarrow \mathcal{AV}^{\Gamma}(\pi_\nu) = \begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline \end{array} \text{———} \begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline - & + \\ \hline \end{array} \text{———} \begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline - & + \\ \hline \end{array} \text{———} \begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline - & + \\ \hline \end{array}$$

Maximally connected subgraph $\cdots$ connected compnts of $\Gamma_{\mathrm{K}}(\mathcal{O})$

Keep considering type AIII
$G = \mathrm{GL}_n \supset K = \mathrm{GL}_p \times \mathrm{GL}_q \;\; (n = p + q) \longleftrightarrow G_{\mathbb{R}} = \mathrm{U}(p, q)$

Theorem (type AIII)

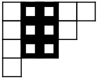
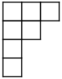
$\mathcal{O} \subset \mathfrak{g}$: *nilpotent G-orbit*
$\forall Z \subset \Gamma_{\mathrm{K}}(\mathcal{O})$: *conn component*
$\Longrightarrow \exists X$: *HC-module s.t.* $Z = \mathcal{AV}^{\Gamma}(X)$

- How to produce such HC-module X?

Key fact due to Barbasch-Vogan

$\forall\, \mathbb{O}$: K-orbit in $\mathfrak{s}$
$\exists\, X_{\mathbb{O}}$: irred derived functor module *s.t.* $\mathcal{AV}(X_{\mathbb{O}}) = \overline{\mathbb{O}}$

Recall λ and $\lambda' \rightsquigarrow$ remove two successive columns of the same length h from λ to get λ'

$\lambda =$  $\longrightarrow \lambda' =$ 

Put $\quad n' = n - 2h, \quad (p', q') = (p - h, q - h)$

$$(G', K') = (GL_{n'}, GL_{p'} \times GL_{q'})$$

$P_{\mathbb{R}}$: $\mathbb{R}$-psg of $G_{\mathbb{R}} = U(p, q)$ s.t.

$$P_{\mathbb{R}} \simeq (U(p', q') \times GL_h(\mathbb{C})) \ltimes N_{\mathbb{R}}$$

- Graph induction brings

$$\text{conn comp of } \Gamma_{K'}(\mathcal{O}_{\lambda'}) \to \text{conn comp of } \Gamma_K(\mathcal{O}_\lambda)$$

- Parabolic induction brings

$$\text{ass var of } (\mathfrak{g}', K')\text{-module} \to \text{ass var of } (\mathfrak{g}, K)\text{-module}$$

In fact they match up!

$G_{\mathbb{R}} = U(p, q)$
$P_{\mathbb{R}} \simeq (U(p', q') \times GL_h(\mathbb{C})) \ltimes N_{\mathbb{R}}$: psg of $G_{\mathbb{R}}$

Theorem

X' : HC-module for $(\mathfrak{g}', K')$-module
$X(\nu) := \operatorname{ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}} X' \otimes |\det|^{\nu}$
Then $\mathcal{AV}^{\Gamma}(X(\nu)) = \text{g-ind}\, \mathcal{AV}^{\Gamma}(X')$ holds, i.e. we have

$$\mathcal{AV}^{\Gamma}(\operatorname{ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(X' \otimes |\det|^{\nu})) = \text{g-ind}_{(G',K')}^{(G,K)}(\mathcal{AV}^{\Gamma}(X'))$$

Starting from

- totally disconnected graph and
- an irred HC-module $X_{\mathbb{O}}$ attached to single $\mathbb{O}$

we can thus construct

- X : HC-module with $\qquad \mathcal{AV}^{\Gamma}(X) = (\text{conn comp of } \Gamma_K(\mathcal{O}))$