Unitary Realization of Cohomologically Induced Modules

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Partially based on joint work with Roger Zierau

Introduction

Problem:

Describe the unitary globalization of a class of cohomologically induced modules,i.e., build a Hilbert space on which each $g \in G_{\mathbb{R}}$ acts by a unitary operator and the resulting action is irreducible.

Method:

Orbit method; build the Hilbert space "from a co-adjoint orbit".

Goal of the lecture:

Describe the status of the problem for elliptic orbits.

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Describe the status of the problem for elliptic orbits.

Set up and Notation:

1. $G_{\mathbb{R}}$ is a connected semisimple Lie group with maximal compact subgroup $K_{\mathbb{R}}$. We let Θ be the corresponding Cartan involution.
2. We write $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}})$; and we let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_{\mathbb{R}}$.
3. G stands for the complexification of $G_{\mathbb{R}}$.
4. θ, the differential of Θ, induces splittings $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
5. $G_{\mathbb{R}}$ acts on $\mathfrak{g}_{\mathbb{R}}^*$ via the co-adjoint rep. For $\lambda \in \mathfrak{g}_{\mathbb{R}}^*$ let

$$L_{\mathbb{R}} = \mathrm{Stab}_{G_{\mathbb{R}}}(\lambda) = \{g \in G_{\mathbb{R}} : \mathrm{Ad}^*(g)\lambda = \lambda\}$$
$$\mathcal{O} = G_{\mathbb{R}} \cdot \lambda \equiv G_{\mathbb{R}}/L_{\mathbb{R}}.$$

6. In this talk we will work with Elliptic co-adjoint orbits i.e., we let $\mathfrak{t}_0$ be a compact torus and choose $\lambda \in t_0^*$. (Note that λ is semisimple with purely imaginary eigenvalues.)

Co-adjoint orbits

We identify $\mathfrak{g}_{\mathbb{R}}^* \simeq \mathfrak{g}_{\mathbb{R}}$ via the Killing form. Observe that $\lambda \in \mathfrak{g}_{\mathbb{R}}^* \leftrightarrow X_\lambda \in \mathfrak{g}_{\mathbb{R}}$.

- The element X_λ defines a complex θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ by means of

$$\begin{aligned} \mathfrak{l} &= \{X \in \mathfrak{g} : \mathrm{ad}(iX_\lambda)X = 0\} \\ \mathfrak{u} &= \{X \in \mathfrak{g} : \mathrm{ad}(iX_\lambda)X = r\, X \text{ with} r > 0\}. \end{aligned}$$

  The splitting $T_e(G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}} \simeq \mathfrak{g}/\mathfrak{l} \simeq \mathfrak{u} \oplus \overline{\mathfrak{u}}$ induces an almost complex structure on $G_{\mathbb{R}}/L_{\mathbb{R}}$, i.e.

$$\begin{aligned} T(G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}} &= T^{1,0}(G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}} \oplus T^{0,1}(G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}} \\ &= \left(G_{\mathbb{R}} \times_{L_{\mathbb{R}}} \overline{\mathfrak{u}}\right) \oplus \left(G_{\mathbb{R}} \times_{L_{\mathbb{R}}} \mathfrak{u}\right) \end{aligned}$$

  that is integrable. Hence, $G_{\mathbb{R}}/L_{\mathbb{R}}$ is a complex manifold.

- When $L_{\mathbb{R}}$ is compact $T(G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}}$ admits an invariant hermitian positive metric. [Identify $T_e(G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}} \simeq \mathfrak{u} \oplus \overline{\mathfrak{u}}$. For $X, Y \in \mathfrak{u}$ define $h(X, Y) = -B(X, \overline{\theta Y})$ where B is the Killing form. h is $L_{\mathbb{R}}$-invariant and induces the positive definite hermitian form on $T(G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}}$.]

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How to attach a representation to $G_{\mathbb{R}}/L_{\mathbb{R}} \simeq G_{\mathbb{R}} \cdot \lambda$ when $G_{\mathbb{R}}$ is compact?

Lemma:

Let $G_{\mathbb{R}}/L_{\mathbb{R}}$ be a manifold that admits an invariant complex structure induced by a complex polarization $\mathfrak{q}$. The space of holomorphic homogeneous bundles over $G_{\mathbb{R}}/L_{\mathbb{R}}$ is in bijection with the space of $(L_{\mathbb{R}}, \mathfrak{q})$-modules.

- Extend the $L_{\mathbb{R}}$-module (E, π) to a $\mathfrak{q}$-module by making $\mathfrak{u}$ act trivially. The bundle attached to (E, π) is

$$\mathcal{E} = G_{\mathbb{R}} \times_{L_{\mathbb{R}}} E \simeq G_{\mathbb{R}} \times E/\sim \text{ where } (g\,\ell, e) \sim (g, \pi(\ell)e).$$

- If $U \subset G_{\mathbb{R}}/L_{\mathbb{R}}$ is open, then the sheaf of germs of holomorphic sections is determined by

$$\mathcal{O}(\mathcal{E})(U) = \{f : \tilde{U} \subset G_{\mathbb{R}} \mapsto E : f(g\,\ell) = \pi(\ell)^{-1} f(g) \text{ and } \pi(\xi)(f(g)) + (r(\xi) \cdot f)(g) = 0 \text{ for all } \xi \in \mathfrak{u}\}$$

where r stands for right action.

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Borel-Weil Theorem

Let $G_{\mathbb{R}} \cdot \lambda \simeq G_{\mathbb{R}}/L_{\mathbb{R}}$ be a co-adjoint orbit with complex polarization $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Fix $\mathfrak{h} \subset \mathfrak{l}$ a Cartan subalgebra and fix $\Delta(\mathfrak{l})^+$. Let $\chi \in \mathfrak{h}^*$ is dominant for $\Delta(\mathfrak{l})^+$ and let E_χ be the $L_{\mathbb{R}}$-rep. with highest weight χ. Let $\Gamma(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{E}_\chi)$ be the space of holomorphic sections of the corresponding bundle with its $G_{\mathbb{R}}$-action. If $\chi + \rho$ is regular and

$$\langle \chi + \rho, \alpha \rangle < 0 \text{ for all } \alpha \in \Delta(\mathfrak{u}),$$

then $\Gamma(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{E}_\chi)$ is an irreducible rep. of $G_{\mathbb{R}}$.
All irreducible rep. of $G_{\mathbb{R}}$ arise this way.

Relevant to the proof of the Theorem are:

- The classification of irred. rep. of a compact group (Theorem of the highest weight) and
- Frobenious reciprocity $\mathrm{Hom}_{G_{\mathbb{R}}}(V, \Gamma(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{E}_\chi)) \simeq \mathrm{Hom}_{L_{\mathbb{R}}}(V, E)$.
- Unitarity essentially follows from the fact that continuous functions on a compact manifold are L^2. We'll get back to this as we need to consider more general situations.

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Examples of Elliptic co-adjoint orbits

$G_{\mathbb{R}} = U(p,q)$ in the usual realization. $t = \mathrm{diag}(x_1, \ldots, x_p) \times \mathrm{diag}(y_1, \ldots, y_q)$.

- If $\lambda = i\,\mathrm{diag}(1, \ldots, 1)) \times \mathrm{diag}(0, \ldots, 0)$, then $D = G_{\mathbb{R}} \cdot \lambda = U(p,q)/U(p) \times U(q) = G_{\mathbb{R}}/K_{\mathbb{R}}$.
  This particular elliptic orbit is a Stein manifold and $T(D)_{\mathbb{C}}$ admits an hermitian positive metric.

- If $\lambda = i\,\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)) \times \mathrm{diag}(0, \ldots, 0)$, then $D = G_{\mathbb{R}} \cdot \lambda = U(p,q)/U(r) \times U(p-r,q) \supset$
  $U(p) \times U(q)/(U(r) \times U(p-r) \times U(q))$ as a compact subvariety.
  D is not Stein.
  This orbit has the particular property that both fibrations
  $G_{\mathbb{R}}/(L_{\mathbb{R}} \cap K_{\mathbb{R}}) \to G_{\mathbb{R}}/K_{\mathbb{R}}$ and $G_{\mathbb{R}}/(L_{\mathbb{R}} \cap K_{\mathbb{R}}) \to G_{\mathbb{R}}/L_{\mathbb{R}}$ are holomorphic.

- If $\lambda = \mathrm{diag}(a, \ldots a, b, \ldots b) \times \mathrm{diag}(a, \ldots a, \ldots, a, b, \ldots b)$, then $D = G_{\mathbb{R}} \cdot \lambda = U(p,q)/U(r,s) \times U(p-r, q-s)$.

How to attach a representation to an elliptic orbit?

Theorem (Harish-Chandra)

Assume that $G_{\mathbb{R}}$ is of hermitian type. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and write the $K_{\mathbb{R}}$ invariant splitting of $\mathfrak{p}$ as $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ Let $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$. Endow $G_{\mathbb{R}}/K_{\mathbb{R}}$ with the complex structure associated to $\mathfrak{q}$ so that $T^{0,1}(G_{\mathbb{R}}/K_{\mathbb{R}}) \simeq \mathfrak{p}^+$. Choose $\mathfrak{h} \subset \mathfrak{k}$ and $\Delta^+ = \Delta^+(\mathfrak{k}) \cup \Delta(\mathfrak{p}^+)$. If $\langle \chi + \rho, \alpha \rangle > 0$ for all $\alpha \in \Delta^+$, then the space of L^2-holomorphic sections $\Gamma^{\text{holo.}}(G_{\mathbb{R}}/K_{\mathbb{R}}, \mathcal{E}_\chi)$ is an irreducible unitary representation of $G_{\mathbb{R}}$.

Remark:

- These are the holomorphic discrete series rep. Not all discrete series arise this way.
- To obtain realizations of a larger class of representations we need to consider higher degree cohomology. (Sheaf cohomology $H^*(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{O}(\mathcal{E}))$.)

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Dolbeault cohomology

Assume that $\mathcal{V} \to M$ is a holomorphic vector bundle over a complex manifold. Let $\mathcal{E}^{p,q}(\mathcal{V})$ be the space of $\mathcal{V}$-valued diff. forms of type (p,q), i.e. the space of smooth sections of $\mathcal{V} \otimes \wedge^p(T^{*(1,0)}(M)) \otimes \wedge^q(T^{*(0,1)}(M))$.

The Dolbeault operator $\overline{\partial} : \mathcal{E}^{p,q}(\mathcal{V}) \to \mathcal{E}^{p,q+1}(\mathcal{V})$ is defined as follows. Let $X_1, \ldots, X_p$ be holomorphic vector fields and let $\overline{Y}_1, \ldots, \overline{Y}_q$ be antiholomrphic vector fields.

$$\overline{\partial}\omega(\overline{Y}, \overline{Y}_1, \ldots, \overline{Y}_q) = \sum_1^p (-1)^{i+p} \overline{Y}_i \cdot \omega(\overline{Y}, \overline{Y}_1 \ldots \widehat{\overline{Y}}_i, \ldots, \overline{Y}_q)$$
$$+ \sum_{i<j} (-1)^{i+j+p}\, \omega(\overline{Y}, [\overline{Y}_i, \overline{Y}_j], \overline{Y}_1 \ldots \widehat{\overline{Y}}_i, \ldots, \widehat{\overline{Y}}_j, \ldots, \overline{Y}_q).$$

Fact: $\overline{\partial}^2 = 0$.

Definition:

Let M be a complex manifold and let $\mathcal{V}$ be a holomorphic vector bundle on M. The (p,q)-Dolbeault cohomology of M with coefficients in $\mathcal{V}$ is

$$H^{p,q}(M, \mathcal{V}) = \text{kernel of } \overline{\partial} \text{ on } \mathcal{E}^{p,q}(\mathcal{V}) / \text{image of } \overline{\partial} \text{ from } \mathcal{E}^{p,q-1}(\mathcal{V}).$$

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Theorem (Hon-Wai Wong)

Let V be a finite dimensional $(\mathfrak{q}, L_{\mathbb{R}})$-module. Let $\mathcal{V}$ be the G-equivariant holomorphic vector bundle over $G_{\mathbb{R}}/L_{\mathbb{R}}$ attached to V. Then,

1. Each cohomology space $H^{p,q}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{V})$ carries a smooth admissible rep. of $G_{\mathbb{R}}$.
2. Under some positivity assumption on the highest weight of V, $H^{n,q}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{V}) = 0$ unless $q = s = \dim(K_{\mathbb{R}}/K_{\mathbb{R}} \cap L_{\mathbb{R}})$.
3. Under some positivity assumption on the highest weight of V, $H^{n,s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{V})$ is non-zero and irreducible.
4. $H^{n,s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{V})$ is the maximal globalization of a cohomologically induced $(\mathfrak{g}, K)$-module.

Remark: A substantial achievement attained by Hon-Wai was to show that the image of $\overline{\partial}$ is closed. Thus, the cohomology admits a Hausdorff topology. Twenty years earlier W. Schmid proved much of this theorem under the assumption $\operatorname{rank} G_{\mathbb{R}} = \operatorname{rank} K_{\mathbb{R}}$ and $L_{\mathbb{R}}$ is compact.

Specialization to $G_{\mathbb{R}}$ connected and compact

- Let $G_{\mathbb{R}}/L_{\mathbb{R}}$ be an elliptic orbit with complex polarization $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$.
- Let $\mathfrak{h} \subset \mathfrak{l}$ be a Cartan subalgebra of $\mathfrak{g}$ and choose a positive system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ so that $\Delta(\mathfrak{u}) \subset \Delta^+(\mathfrak{g}, \mathfrak{h})$.

Theorem (Borel-Weil-Bott)

Let V^λ be an irreducible representation of $L_{\mathbb{R}}$ with highest weight λ. Assume that λ is Δ^+-dominant. Then

$$H^{0,j}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{V}^\lambda \otimes \wedge^{\mathrm{top}}\mathfrak{u}) = \begin{cases} 0 & \text{if } j \neq \dim_{\mathbb{C}}(G_{\mathbb{R}}/L_{\mathbb{R}}) \\ F^\lambda & \text{if } j = \dim_{\mathbb{C}}(G_{\mathbb{R}}/L_{\mathbb{R}}) \end{cases}$$

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Hodge Theory

- Recall that our compact complex manifold $G_{\mathbb{R}}/L_{\mathbb{R}}$ admits an invariant hermitian metric. Hence we have an orthocomplementation operator
$$\overline{\star}: \mathcal{V} \otimes \wedge^p(T^*((G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}}) \to \mathcal{V}^h \otimes \wedge^{2n-p}(T^*((G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}})$$
where $\dim_{\mathbb{R}}(G_{\mathbb{R}}/L_{\mathbb{R}}) = 2n$.
- If $\mathcal{E}^m(\mathcal{V})$ is the space of $\mathcal{V}$-values m-forms, then $\mathcal{E}^m(\mathcal{V}) = \oplus_{p+q=m}\mathcal{E}^{p,q}(\mathcal{V})$. We have
$$\overline{\star}: \mathcal{E}^{p,q} \to \mathcal{E}^{n-p,n-q}.$$
- There is a well-defined positive definite inner product on the space of (p,q)-forms. If $\eta_1, \eta_2 \in \mathcal{E}^{p,q}$, then
$$\langle \eta_1, \eta_2 \rangle = \int_{G_{\mathbb{R}}/L_{\mathbb{R}}} \eta_1 \wedge \overline{\star}\eta_2.$$
- $\overline{\partial}$ has a formal adjoint operator $\overline{\partial}^*$ that is invariant and the Laplace-Beltrami operator $\square = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is both elliptic and self-adjoint.
- Since $\square$ is elliptic, $\square\psi = 0 \Rightarrow \psi$ is smooth.
- Since $\langle , \rangle$ is a metric $\square\psi = 0 \Rightarrow \overline{\partial}\psi = 0$ and $\overline{\partial}^*\psi = 0$.

Definition:

$$L^2_s(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_{\chi+2\delta(\mathfrak{u})}) = \{ \text{ square integrable } (0,s) \text{ forms}\}$$
$$\mathcal{H}^2_s = \text{ Kernel}(\Box) \cap L^2_s(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_{\chi+2\delta(\mathfrak{u})}).$$

Theorem (Hodge)

Keep the hypothesis of the Borel-Weil-Bott theorem. The map

$$\begin{aligned} \mathcal{H}^2_s &\to H^{0,s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi \otimes \wedge^{top}(\mathfrak{u})) \\ \omega &\mapsto [\omega] \end{aligned}$$

is an isomorphism.

Remark:

1. We know that harmonic forms are closed forms, so the map is well-defined.
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Is it possible to use an analogue construction in the non-compact case?

Lemma:

Let $G_{\mathbb{R}}/L_{\mathbb{R}}$ be an elliptic orbit with complex polarization $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ so that $T^{0,1}(G_{\mathbb{R}}/L_{\mathbb{R}}) \simeq \mathfrak{u}$. Then,

1. $G_{\mathbb{R}}/L_{\mathbb{R}}$ has an invariant (possibly indefinite) hermitian metric. The metric can be defined in terms of the Killing form.
2. If $L_{\mathbb{R}}$ is compact, then $G_{\mathbb{R}}/L_{\mathbb{R}}$ has a positive definite invariant metric.

Proof. Let B be the Killing form on $\mathfrak{g}$. Set

$$\langle \xi, \eta \rangle_{\text{inv}} = B(\xi, \overline{\eta}) \text{ for } \xi, \eta \in \mathfrak{u} \simeq T_e^{0,1}(G_{\mathbb{R}}/L_{\mathbb{R}}).$$

As B is nondegenerate on $\mathfrak{g}$ and on $\mathfrak{l}$, it is nondegenerate on $(\mathfrak{l})^{\perp} = \mathfrak{u} + \overline{\mathfrak{u}}$. Hence $\langle , \rangle_{\text{inv}}$ is a nondegerate hermitian form on $\mathfrak{u}$. Since $\langle , \rangle_{\text{inv}}$ is $L_{\mathbb{R}}$-invariant, there is a well-defined metric on $T^*(G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}}$. The signature is (a, b) with $a = \dim(\mathfrak{p} \cap \mathfrak{l}^{\perp})$ and $b = \dim(\mathfrak{k} \cap \mathfrak{l}^{\perp})$.
When $L_{\mathbb{R}}$ is compact, $-B(\xi, \theta\overline{\eta})$ is $L_{\mathbb{R}}$-invariant and induces a positive metric $\langle , \rangle_{\text{pos}}$ on $T(G_{\mathbb{R}}/L_{\mathbb{R}})_{\mathbb{C}}$.

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Strongly Harmonic Forms

- The formal adjoint of $\overline{\partial}$ satisfies

$$\langle \overline{\partial}\eta_1, \eta_2 \rangle_{\mathrm{inv}} = \int_{G_\mathbb{R}/L_\mathbb{R}} \langle \overline{\partial}\eta_1(x), \eta_2(x) \rangle_{\mathrm{inv}} = \int_{G_\mathbb{R}/L_\mathbb{R}} \langle \eta_1(x), \overline{\partial}^* \eta_2(x) \rangle_{\mathrm{inv}} = \langle \eta_1, \overline{\partial}^* \eta_2 \rangle_{\mathrm{inv}}$$

for compactly supported forms.

- The Laplace-Beltrami operator, $\square = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, is usually not elliptic. Moreover, $\square\eta = 0 \not\Rightarrow \overline{\partial}\eta = 0$.

Definition:

A $(0, p)$-form η on $G_\mathbb{R}/L_\mathbb{R}$ is strongly harmonic if $\overline{\partial}\eta = 0$ and $\overline{\partial}^*\eta = 0$. We let $\mathcal{H}_p(G_\mathbb{R}/L_\mathbb{R}, \mathcal{L}_\chi)$ denote the space of strongly harmonic forms of type $(0, p)$.

The Hilbert Space

Begin with the positive definite hermitian form $\langle \xi, \eta \rangle_{\text{pos}} = -B(\xi, \theta\overline{\eta})$ on $\mathfrak{u} + \overline{\mathfrak{u}} \simeq (T_{eL_\mathbb{R}}(G_\mathbb{R}/L_\mathbb{R}))_\mathbb{C}$. As $\langle , \rangle_{\text{pos}}$ is not $L_\mathbb{R}$-invariant we can not simply translate to an arbitrary tangent space by simply defining $\langle \ell_x \xi, \ell_x \eta \rangle_{\text{pos}} = \langle \xi, \eta \rangle_{\text{pos}}$.

Proposition:

Write $\mathfrak{l}_\mathbb{R}^\perp = \{\xi \in \mathfrak{g}_\mathbb{R} : B(\xi, \eta) = 0 \text{ for all } \eta \in \mathfrak{l}_\mathbb{R}\}$. Then

$$G_\mathbb{R} = K_\mathbb{R} \exp(\mathfrak{p}_\mathbb{R} \cap \mathfrak{l}_\mathbb{R}^\perp) \exp(\mathfrak{p}_\mathbb{R} \cap \mathfrak{l}_\mathbb{R}).$$

The positive metric on the tangent space at $xL_\mathbb{R}$ given by

$$\langle \ell_{\kappa(x)\exp(X(x))}(\xi), \ell_{\kappa(x)\exp(X(x))}(\eta) \rangle_{\text{pos}}$$

is well-defined and $K_\mathbb{R}$-invariant.

Definition:

For smooth compactly supported forms set

$$\langle \omega_1, \omega_2 \rangle_{\text{pos,glob}} = \int_{G_\mathbb{R}/L_\mathbb{R}} \langle \omega_1(\kappa(x)\exp(x)), \omega_2(\kappa(x)\exp(x)) \rangle_{\text{pos}}.$$

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Indefinite Quantization

Define $L_2^{0,p}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi)$: completion of compactly supported smooth $(0,p)$-forms with respect to $\langle , \rangle_{\text{pos,glob}}$.

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A relevant fact is that the hermitian form $\langle , \rangle_{\text{inv,glob}}$ is bounded above by $\langle , \rangle_{\text{pos,glob}}$. Hence, $\mathcal{H}_2^s(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi)$ has an invariant possible indefinite hermitian form.

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Indefinite Harmonic Theory (Rawnsley-Schmid-Wolf)

There is a satisfactory quantization procedure if the following hold.

1. $\mathcal{H}_2^s$ is nonzero.
2. The image of the quotient map $q : \mathcal{H}_2^s(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi) \to H^{0,s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi)$ contains all K-finite cohomology classes.
3. The kernel of q coincides with the radical of $\langle , \rangle_{\text{inv,glob}}$. .
4. $\langle , \rangle_{\text{inv, glob}}$ is positive semidefinite on $\mathcal{H}_2^s$.
5. $$\overline{\mathcal{H}_2^s} = \mathcal{H}_2^s / \{\text{nullspace of } \langle , \rangle_{\text{inv,glob}}\}$$
   is complete with respect to $\langle , \rangle_{\text{inv,glob}}$.

Theorem (Rawnsley-Schmid-Wolf)

Assume that $G_\mathbb{R}/L_\mathbb{R}$ is semisimple symmetric and that the fibrations $G_\mathbb{R}/(L_\mathbb{R} \cap K_\mathbb{R}) \to G_\mathbb{R}/K_\mathbb{R}$ and $G_\mathbb{R}/(L_\mathbb{R} \cap K_\mathbb{R}) \to G_\mathbb{R}/L_\mathbb{R}$ are holomorphic. Under some positivity assumption on χ the quantization procedure holds and $(\mathcal{H}_2^s, \langle , \rangle_{\text{inv, glob}})$ is the unitary globalization of $H^{0,s}(G_\mathbb{R}/L_R, \mathcal{L}_\chi)_{K-\text{finite}}$.

- The holomorphicity assumptions imply that the $G_\mathbb{R}$-rep. on $H^{0,s}(G_\mathbb{R}/L_R, \mathcal{L}_\chi)$ is a highest weight module (not every highest weight module arise this way.)
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Theorem

Let $G_{\mathbb{R}}/L_{\mathbb{R}}$ be an elliptic co-adjoint orbit. Under some positivity assumptions on χ, every K-finite cohomology class in $H^{0,s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi \otimes \wedge^{\text{top}}(\mathfrak{u}))$ admits a strongly harmonic representative.

Remarks:

The representatives are explicitly given by integral formulae. This was done in the 90's; the equal rank case jointly with Knapp and Zierau.
The idea of the construction comes from Complex Analysis.

Recall that for the unit ball in $\mathbb{R}^n$, the Poisson transform maps functions on the boundary of the ball to harmonic functions on the ball. For the ball in $\mathbb{C}^n$ the Szegö map transforms functions on the boundary to holomorphic functions on the ball. For Riemannian Symmetric spaces the Poisson transform produces harmonic functions with respect to the Euclidean metric.

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- We place $H^{0,s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi \otimes \wedge^{\text{top}}(\mathfrak{u}))_{K-\text{finite}}$ as the unique irred. quotient in an induced module $\text{ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(\delta_M, \rho_L, 1)$. We define a kind of Szegö map from $\text{ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(\delta_M, \rho_L, 1)$ to the space of strongly harmonic forms. When passing to cohomology the map is onto $H^{0,s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi \otimes \wedge^{\text{top}}(\mathfrak{u}))_{K-\text{finite}}$. We write the strongly harmonic representative as $\mathcal{S}f$ with $f \in \text{ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(\delta_M, \rho_L, 1)$.
- A similar construction for producing strongly harmonic forms holds for representation in the discrete series.

Theorem (B.-Zierau)

Assume $L_{\mathbb{R}}$ is the fixed-point of an involution and that real ranks of $G_{\mathbb{R}}$ and $L_{\mathbb{R}}$ coincide. Under some positivity assumption of χ, the invariant form $\langle , \rangle_{\text{inv,glob}}$ is not identically zero on $\mathcal{H}_2^s(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi)$. The completion of $\overline{\mathcal{H}_2^s} = \mathcal{H}_2^s/\{\text{ nullspace of } \langle , \rangle_{\text{inv,glob}}\}$ with respect to $\langle , \rangle_{\text{inv,glob}}$ is the unitarity globalization of $H^{n,s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi)_{K-\text{finite}}$. In particular, the image of $\mathcal{S}$ consists of strongly harmonic square integrable forms.

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1. Each K-finite cohomology class admits a representative of the form $\mathcal{S}f$ with $f \in \mathrm{ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(\delta_M, \rho_L, 1)$.

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Remarks:

- As $\langle \mathcal{S}f, \mathcal{S}f\rangle_{\text{inv}} \in L^1(G_{\mathbb{R}}/L_{\mathbb{R}})$, we have a well defined $\langle \mathcal{S}f, \mathcal{S}f\rangle_{\text{inv,glob}}$ without the need of an auxiliary metric.
- The comparison with the hermitian form on the Langlands quotient suggests that $\langle , \rangle_{\text{inv,glob}}$ is the metric that gives the unitary globalization.
- If C is non-zero the theorem says that $\{K\text{-finite image of } \mathcal{S}\}/\text{nullspace of } \langle , \rangle_{\text{inv,glob}}$ admits a definite hermitian form. The completion is the unitary globalization of the underlying $(\mathfrak{g}, K)$-module.
- The constant C is related to the realization of the discrete series of M. [When $L_{\mathbb{R}}$ is compact, $G_{\mathbb{R}}/L_{\mathbb{R}}$ admits two invariant metrics , $\langle , \rangle_{\text{inv,glob}}$ and $\langle , \rangle_{\text{pos,glob}}$. The constant C comes from comparing these metrics.] Schmid used $\langle , \rangle_{\text{pos,glob}}$ to realize rep. in the discrete series as a L^2-harmonic space. *We need an indefinite quatization theory for rep. in the discrete series.*

A different approach

- By a result of Harish-Chandra every admissible $(\mathfrak{g}, K)$-module that is finitely generated is the Harish-Chandra module of a representation (π, V) of $G_{\mathbb{R}}$. There is no "uniqueness" in this statement. Indeed, by work of Casselman-Wallach and Schmid given (π, V) admissible $G_{\mathbb{R}}$-rep. on a reflexive Banach space, then $(V^{\omega}, \pi^{\omega}), (V^{\infty}, \pi^{\infty}), (V^{-\infty}, \pi^{-\infty}), (V^{-\omega}, \pi^{-\omega})$ (analytic, smooth, distribution, hyperfunctions) are smooth rep. all infinitesimally equivalent to π.

- A different approach to finding the unitary globalization of $H^{n,s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_{\chi})_{K-\text{finite}}$ is to identify the minimal globalization and to define an invariant positive definite hermitian form on it.

- The minimal globalization of $H^{n,s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_{\chi})_{K-\text{finite}}$ is the topological dual of the maximal globalization i.e Dolbeault cohomology space. Tim Bratten proved that the minimal globalization is given by $H_c^{0,n-s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_{\chi}^{*})$ compactly supported cohomology.

- The space of invariant hermitian forms on $H_c^{0,n-s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi^*)$ is the space of $G_{\mathbb{R}}$-intertwining maps from $H_c^{0,n-s}(G_{\mathbb{R}}/L_{\mathbb{R}}, \mathcal{L}_\chi^*)$ to its hermitian dual, $H^{n,s}(\overline{G_{\mathbb{R}}/L_{\mathbb{R}}}, \mathcal{L}_\chi^h)$.
  Vogan identifies this space with the $diag(G_{\mathbb{R}} \times G_{\mathbb{R}})$-invariant real forms on $H^{(n,n)(s,s)}(G_{\mathbb{R}}/L_{\mathbb{R}} \times \overline{G_{\mathbb{R}}/L_{\mathbb{R}}}, \mathcal{F}_\chi)$ with $\mathcal{F}$ induced by the invariant hermitian pairings on $\mathcal{L}_\chi$. Under our assumptions this is a one dimensional space.
- Vogan propose to describe geometrically this diag $(G_{\mathbb{R}} \times G_{\mathbb{R}})$-invariant real form on $H^{(n,n)(s,s)}(G_{\mathbb{R}}/L_{\mathbb{R}} \times \overline{G_{\mathbb{R}}/L_{\mathbb{R}}}, \mathcal{F}_\chi)$. This form is to be thought of as a reproducing kernel.