## SINGULARITIES OF NILPOTENT VARIETIES

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G =connected complex reductive algebraic group

 $\mathfrak{g} = \operatorname{Lie} G$ 

 $e \in \mathfrak{g}$ nil<br/>potent

 $\mathcal{O} = G \cdot e$ , nilpotent orbit

 $\overline{\mathcal{O}}$  = nilpotent variety

**Question**: When is  $X = \overline{\mathcal{O}}$  rationally smooth (i.e., when does it have the same cohomology as a smooth variety, nonvanishing in only one degree?

Motivation: B = Borel subgroup of G

G/B =flag variety

B has finitely many orbits in G/B, indexed by Weyl group W.

Corresponding question of when  $\overline{\mathcal{O}}$  is rationally smooth has many consequences in representation theory; it is computed by the famous Kazhdan-Lusztig polynomials, which also compute multiplicities of composition factors in Verma modules, an important class of modules of finite length. Smoothness of these orbits, by contrast, has no such representation-theoretic interpretation.

**Real setting**: Look at *K*-orbits in G/B, K the fixed points of some involution  $\theta$ . Here again there is a family of polynomials computing simultaneously whether a *K*-orbit closure is rationally smooth **and** the character of the irreducible  $\mathfrak{g}, K$  module corresponding to the orbit in terms of standard characters.

In our case I know of no representation-theoretic interpretation of the rational smoothness or singularity of a nilpotent variety, but the geometry of  $\mathfrak{g}/G$  is closely related to that of G/B; it is no coincidence that G has finitely many orbits on both the nullcone  $\mathcal{N}$  and the product  $G/B \times G/B$ .

**Theorem.** If  $\mathcal{O}$  is neither principal nor 0, then  $\overline{\mathcal{O}}$  is rationally singular, except in two cases:  $\mathcal{O}_{min}$  in type  $C_n$  or  $G_2$ .

This may be rather surprising, since the full nullcone  $\mathcal{N}$ , the closure of the principal orbit, is always rationally smooth, by an old result of Borho and MacPherson (one of the first applications of intersection cohomology). On the other hand, the singular locus of any orbit closure  $\overline{\mathcal{O}}$  is always the full boundary  $\partial \overline{\mathcal{O}}$ , by a recent result of Kaledin.

The proof of the theorem is based on techniques of Brion:

Brion: Let X be an algebraic variety with an action of a torus T fixing a point x of X. Then

 $X \text{ rationally smooth at } x \Rightarrow \dim_x X - \dim_x X^T = \sum_{T' \subset T \atop \operatorname{codim} T' = 1} (\dim_x X^{T'} - \dim_x X^T)$ 

and

X rationally smooth at 
$$x \Rightarrow X^{T'}$$
 rationally smooth at x

In our setting T = maximal torus of G, x = 0. If  $\mathcal{O} = \mathcal{O}_{\text{prin}}$ , then  $X^T = 0$  and  $X^{T'} \neq 0$  exactly when  $T' = T_{\alpha}$ , the subtorus corresponding to a root  $\alpha$  of T in  $\mathfrak{g}$ ; in this case it has dimension 2. In general the RHS is the same as for  $\mathcal{O}_{\text{prin}}$ , provided that  $\overline{\mathcal{O}}$  strictly contains  $\overline{\mathcal{O}}_{\min}$ , but the LHS is strictly less than dim  $\mathcal{N}$ , so equality fails. Same holds if  $\mathcal{O} = \mathcal{O}_{\min}$ , except in types  $C_n, G_2$ .

What about sufficient conditions for rational smoothness, in general?

Brion: With notation as the necessary condition state above, suppose that x is an *attractive* T-fixed point of X, so that all weights in the tangent space  $T_x X$  lie in an open half-space. Then

$$\dim_x X \le \sum_{T'} \dim_x X^{T'}$$

in general, and X is rationally smooth at x if and only if

- A punctured neighborhood of x in X is rationally smooth
- $X^{T'}$  is rationally smooth at x for all T' of codimension one in T
- $\dim_x X = \sum_{T'} \dim_x X^{T'}$

The same condition holds more generally if all nonzero weights of T in the tangent space  $T_x X$  lie in an open half space, provided that we subtract  $\dim_x X^T$  from both  $\dim_x X$  and from  $\dim_x X^{T'}$  in the last condition. Here 0 is not necessarily attractive for the T-action on  $\overline{\mathcal{O}}$ , but if we let  $G \times \mathbb{C}^*$  act on  $\overline{\mathcal{O}}$  with  $\mathbb{C}^*$  by scalar multiplication, then 0 is attractive for the  $T \times \mathbb{C}^*$  action.

The first hypothesis above is hard to check in general, but for  $\overline{\mathcal{O}}_{\min}$  there is only one singular point 0, so that  $\mathcal{O}_{\min}$  is the desired punctured neighborhood.

To test the first hypothesis for rational smoothness in general, we have the following:

Let  $S \subset X$  be a slice, so that there is an algebraic group H acting on X, x is isolated in  $S \cap Hx, S$  is stable under a maximal torus T of the isotropy group  $H_x$  and the map  $(h, s) \to hs$  is smooth at (1, x) (its differential is surjective on the tangent spaces). Then, if x is attractive for the T-action on S:

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## S rationally smooth at $x \Leftrightarrow X$ rationally smooth at x

Start with the nullcone  $\mathcal{N}_1$  for  $\mathfrak{sl}_2$ , which identifies with the quadric  $\{(x, y, z) : x^2 + yz = 0\}$ . This is rationally smooth because (0, 0, 0) is its only singular point and we have an action of a 2-torus T on it via  $(a, b) \cdot (x, y, z) = (ax, aby, ab^{-1}z)$ : here two proper subtori fix lines and no other subtorus fixes more than a point. Now we inductively construct a slice  $\mathcal{S}$  for the nullcone  $\mathcal{N}$  of  $\mathfrak{g}$ , using also the adjoint action of U on  $\mathcal{N}, U$  the unipotent radical of a Borel subgroup of G.

Then S is realized as the product of an affine space  $\mathbb{C}^r$  and a variety with only one singular point, at which it is rationally smooth, by Brion's criterion. Hence S and  $\mathcal{N}$  are rationally smooth, recovering Borho-Macpherson's result.

Using these results, we can test rational smoothness at points other than 0. We find that rational singularity, like singularity, is quite prevalent at boundary points, but there are some notable exceptions.

For example, the spherical orbits in type  $C_n$  (those admitting dense suborbits under the action of a Borel subgroup; they correspond to partitions of the form  $2^m 1^{2n-2m}$ ) are such that all points in the next lower orbit  $\mathcal{O}_{2^{m-1},1^{2n-2m+2}}$  are rationally smooth in the closure, but all points in orbits further down are rationally singular. In particular, we recover that  $\overline{\mathcal{O}}_{\min}$  is rationally smooth. To verify this, or more generally to test for rational smoothness at points lying in principal suborbits for Levi factors L of G, we replace the maximal torus T above by the subtorus T' such that  $G^{T'} = L$ .

Consequences for the Springer correspondence: The Springer representation of the Weyl group W attached to a nonprincipal, nonzero orbit and the trivial local system on it *never* has dimension 1, except for the minimal orbit in types  $C_n$  and  $G_2$ . There are similar consequences for other nonzero rationally singular points in other orbit closures.