TITS SYSTEMS, PARABOLIC SUBGROUPS, PARABOLIC SUBALGEBRAS

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In this paper we give a brief survey of the basic results on **B-N** pairs and **parabolic subgroups** as seen in Bourbaki and Humphreys. We also describe two applications of the theory of parabolics subgroups: the Bala-Carter classification of unipotent orbits in simple algebraic groups, and an extension of that theory to the classification of nilpotent orbits in real reductive Lie algebras. The second application is part of the author's doctoral thesis under the supervision of Prof. Donald R. King. (in preparation)

1. Historical Remarks (see Bourbaki and K. Brown)

At the time of the publication of Bruhat's seminal work on the representation of complex Lie groups, in 1954, mathematicians knew how to associate to a Lie group G, a finite reflection group W, called the Weyl group. It is given by $W = N/T$, where T is a maximal torus and N its normalizer. Some were also aware of a certain subgroup $B \subset G$ that plays a fundamental role in the work of Borel. The Bruhat decomposition $G = BWB$ provided a connection between B and W that was not yet known. More precisely he proved that the set $B \setminus G/B$ of double cosets was finite and in 1-1 correspondence with the finite reflection group W.

Soon after, Chevalley generalized the above results for not only the classical groups but also for the exceptional ones. Moreover he worked over an arbitrary field. In a 1962 paper, Tits gave an axiomatic treatment of Chevalley's work. Later, in 1964, he and Matsumoto discovered, independently, that the Weyl group was in fact a Coxeter group. In his work on **Buildings** Tits explained how a group with a BN-pair gave rise to a geometry satisfying his axioms for buildings. Finally, Feit and Higman proved that the only "unusual" Weyl group that can arise from a finite group with with BN-pair is $I_2(8)$, the dihedral group of oder 16.

This theory has been used extensively over the last decades to develop the representation theory of groups. In a recent book Knapp and Vogan use a version of it to study **Cohomological Induction**.

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2. Tits Systems

Let G be a group and B a subgroup of G. Then the group $B \times B$ acts on G by $(b, b').g = bgb'^{-1}$ for $b, b' \in B$ and $g \in G$. The orbits of $B \times B$ in G are the sets BgB. They form a partition of G; the corresponding quotient space is denoted by $B \setminus G/B$.

Assume that G is generated by B and an other subgroup N such that $T = B \cap N$ is normal in N. Let $W = N/T$ and suppose that W is generated by a subset S consisting of involutions (elements of order 2). Then we have

Definition. (G,B,N,S) is a Tits System, provided that the following axioms are satisfied.

T1 . If $\rho \in S, \sigma \in W$ then $\rho B \sigma \subset B\sigma B \cup B\rho\sigma B$

T2 . If $\rho \in S$ then $\rho B \rho \neq B$

W is usually called the **Weyl group** of the Tits system, and B, or any of its conjugates in G, a **Borel subgroup** of G. The cardinal of S is called the **rank** of the system. Observe that any element of W is a class modulo T, hence a subset of G. Therefore the products of the form BwB for $w \in W$ make sense. Moreover for any subset A of W we define $BAB = \bigcup_{w \in A} BwB$.

1. An example

Let $G = GL(2, \mathbb{R})$ be the set of 2×2 invertible real matrices and let (e_1, e_2) be the canonical basis of $\mathbb{R}^2$. Finally Let $B \subset G$ be the group of upper triangular matrices in G. Denote by N the subgroup of G consisting of matrices with exactly one non zero entry in each row and each column. N is usually called the monomial group. Then B stabilizes the one dimensional space generated by e_1 while N is the stabilizer of the lines $\mathbb{R}e_1$ and $\mathbb{R}e_2$. More precisely an element of N permutes the lines $\mathbb{R}e_i$. It follows that N acts as a group of permutations on two letters. Hence we have a surjective homomorphism from N onto $\mathfrak{S}_2$ the symmetric group of 2 letters. The kernel of this homomorphism is exactly the diagonal subgroup of G. It is $T = B \cap N$. Clearly under conjugation N normalizes T and $W = N/T$ can be identified with $\mathfrak{S}_2$. Furthermore we have

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

In other words the group $< B, N >$ contains the set of lower triangular matrices. From linear algebra we know that every invertible matrices decomposes into a product of a lower triangular matrix and an upper triangular one. Hence $G =<$

$B, N >$. Also the above product shows that the axiom T2 is verified. It remains to check T1. Since the type of matrices involved are so simple and the elements of W are of two types ,$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ or $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, it is easy to check T1 directly.

2. Some properties

Using the above notations define $C(w) = BwB$, $w \in W$. Then for w, $w' \in W$ and s, $s' \in S$ we have:

1. $C(e) = BeB = B$, where e is the identity in G.
2. $C(ww') = Bww'B \subset BwBBw'B = C(w).C(w')$
3. $C(w^{-1}) = C(w)^{-1}$, for

$$x \in C(w^{-1}) \iff x = b_1 w b_2 \; b1, b2 \in B \iff$$

$$\iff x^{-1} = b_2^{-1} w b_1^{-1} \iff x^{-1} \in C(w) \iff x \in C(w)^{-1}.$$

4. Axiom T1 implies that $C(s)C(w) = B(sBw)B \subset C(w) \cup C(sw)$. Since $C(sw) \subset (s)C(w)$ and $C(s)C(w)$ is the union of two classes there can only be two possibilities:

$$C(s)C(w) = \begin{cases} C(sw), & if C(w) \not\subseteq C(s)C(w) \\ \\ C(w) \cup C(sw), & if C(w) \subset C(s)C(w) \end{cases}$$

5. Axiom T2 implies that $B \neq C(s)C(s)$. Since $s^2 = e$ we have

$$C(s)C(s) = B \cup C(s).$$

Thus $B \cup C(s)$ is a subgroup of G.

3. Bruhat Decomposition

For $I \subset S$ let $W_I =< I >$. Define $P_I = BW_I B$. Then we have:

Theorem. *If $I \subset S$ Then P_I is a subgroup of G. Furthermore for $\sigma, \sigma' \in W$ $B\sigma B = B\sigma' B$ if and only if $\sigma = \sigma'$.*

Proof. (See Humphreys)

As a consequence of the above theorem, $G = BWB$ for $I = S$.

4. Generators and Relations for W

For the Tits system $G(2, \mathbb{R}, B, N, S)$ described in the preceding example, W was generated by a single element ρ subject to the relation $\rho^2 = e$. If the rank of the system is 2, then W is homomorphic to an abstract group whose presentation is $< \hat{\rho}_1, \hat{\rho}_2 | \hat{\rho}_i^2 = e = (\hat{\rho}_1 \hat{\rho}_2)^m >$. In this case W is dihedral. We assume that S has finite cardinality. By definition a **Coxeter group** is a group with generators $\hat{\rho}_i$, $1 \leq i \leq l$ and defining relations $(\hat{\rho}_i, \hat{\rho}_j)^{m(i,j)} = e$, where $m(i,i) = 1$ and $m(i,j) = m(j,i)$ is a number greater than or equal to 2. Then the following theorem shows that W is actually a Coxeter group.

Theorem. *Let $m(i,j)$ be the order of $\rho_i \rho_j$. Let $\pi : \hat{W} \to W$ be the canonical epimorphism, where $\hat{W}$ is a Coxeter group. Then π is an isomorphism.*

Proof. (See Bourbaki , Humphreys)

The poof of the preceding theorem use the fact that W satisfies an " exchange condition". For $\sigma \in W$ we call an expression $\sigma = \rho_1 \ldots \rho_k$, $\rho_i \in S$, **reduced** if k is as small as possible and we write $l(\sigma) = k$. This is the length of σ relative to S. By convention $l(\sigma) = 0 \iff \sigma = e$. The meaning of the exchange condition is made clear by

lemma. *Let $\sigma \in W$ have reduced expression $\rho_{i(1)} \ldots \rho_{i(t)}$. Suppose that $l(\rho_{i(0)}\sigma) \leq l(\sigma)$. Then, there exists s, $1 \leq s \leq t$, such that $\rho_{i(o)}\rho_{i(1)} \ldots \rho_{i(s-1)} = \rho_{i(1)} \ldots \rho_{i(s)}$.*

Proof. (Humphreys)

5. A Simplicity Criterion

Lemma. *Let H be a normal subgroup of G. Then, there is a partition $S = I \cup J$ such that I, J commute elementwise and $HB = P_I$.*

Proof. (Bourbaki, Humphreys)

Any partition of S into subsets I, J which commute elementwise yields a decomposition of W as a direct product $W_I \times W_J$. If no non trivial decomposition of this sort exists, we say that W is **irreducible**. A simplicity criterion is expressed by

Theorem. *Let W be irreducible, and assume that G is generated by the conjugates of a normal solvable subgroup U of B, while $G = (G, G)$. Then G/Z ,where $Z = \bigcap_{x \in G} xBx^{-1}$, is simple (or trivial).*

Proof. (Humphreys)

6. Parabolic Subgroups

Sometimes a Tits system is written as (G, B, N). This is justified by:

Lemma. *S is precisely the set of those $\sigma \in W$ for which $B \cup B\sigma B$ is a group (so (G, B, N) determine S uniquely), and S is a minimal generating set for W.*

Proof. (Humphreys)

The subgroups of G containing a Borel subgroup are called **parabolic subgroups**. They play a fundamental role in Representation Theory and are classified by:

Theorem.

a) The only subgroups of G containing B are those of the form P_I,$I \subset S$.
b) If P_I is conjugate to P_J, then $P_I = P_J$.
c) $N_G(P_I) = P_I$.
d) If $W_I \subset W_J$, Then $I \subset J$.
e) If $P_I \subset P_J$, Then $I \subset J$.

Proof. (Humphreys)

An obvious consequence of the theorem is that B is self-normalizing in G. Moreover, the theorem shows that the lattice of subgroups of G containing B is isomorphic to the lattice of subsets of S ordered by inclusion, or to the lattice of subgroups W_I of W.

It is a fact [**Humphreys**] that the quadruple (G, B, N, S) is a Tits system, where G is a reductive group, B a Borel subgroup including a maximal torus T, $N = N_G(T)$, W the Weyl group, S the set of simple reflections corresponding to the base, Δ, of the root system determined by B. Observe that the rank of the system is equal to the semisimple rank of G. From now on we will work with that system.

The parabolic subgroups of G containing B (not one of its conjugates) are called **standard parabolic subgroups** of G relative to B. They correspond 1-1 to the 2^l subsets of Δ. Let $\mathfrak{g}$ be the Lie algebra of G. Then the Lie algebra of P_I is $p_I = \mathfrak{k} \oplus \coprod \mathfrak{g}_\alpha$, where $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$, α ranges over some set of roots Θ including the set of positive roots Φ^+ and $\mathfrak{g}_\alpha$ the root space corresponding to α. The Lie algebra of a parabolic subgroup is called a parabolic subalgebra that is a subalgebra containing the Lie algebra of a Borel subgroup. The following theorem tells us how to determine Θ.

Theorem.

a) Each parabolic subgroup of G is conjugate to one and only one subgroup P_I, where $I \subset \Delta$.

b) The roots of P_I relative to T are those in Φ^+ along with those roots in Φ^- which are $\mathbb{Z}$-linear combinations of I.

Proof. (Humphreys)

Example. Let $G = Sl_4$. Then $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$. For $I = \{\alpha_1, \alpha_2\}$ we have

$$p_I = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

1. Levi Decompositions

Let V be the unipotent part of the radical of P_I. Let Ψ be the subsystem of roots spanned by I. Then $p_I = l \oplus v$, where $l == \mathfrak{k} \oplus \coprod_{\alpha \in \Psi} \mathfrak{g}_\alpha$. In fact l is a reductive subalgebra of $\mathfrak{g}$, while v is a nilpotent ideal. Then P_I is the semidirect product $L.V$, where $L = Z_G(S)$ and $S = (\bigcap_{\alpha \in I} ker\alpha)^0$. The corresponding decomposition of P_I as a semidirect product of LV is called a **Levi decomposition**, and we call L a **Levi factor**. In general, such a decomposition exists in an arbitrary connected algebraic group when char 0, and any two Levi factors of a parabolic subgroup P are conjugate under its unipotent radical. However, such decomposition may not exist in cases where the characteristic is prime.

From the above example we deduce that

$$l_{_I} = \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

while,

$$v_{_I} = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

7. Applications

1. The Bala-Carter Classification

Let $\mathfrak{g}_{\mathbb{C}}$ be a semisimple complex Lie algebra and $G_{\mathbb{C}}$ its adjoint group. The Jacobson-Morozov theorem tells us that any nilpotent element X in $\mathfrak{g}_{\mathbb{C}}$ can be embedded in a triple (H, X, Y) of $\mathfrak{g}_{\mathbb{C}}$, where H is semisimple , Y is nilpotent, with the following relations:

$$[H, X] = 2X \ , \ [H, Y] = -2Y \ , \ [X, Y] = H$$

It is a fact that $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{j \epsilon \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^{(j)}$ where $\mathfrak{g}_{\mathbb{C}}^{(j)} = \{Z \epsilon \mathfrak{g}_{\mathbb{C}} | [H, Z] = jZ\}$, and the subalgebra $\mathfrak{q} = \bigoplus_{j \epsilon \mathbb{N}} \mathfrak{g}_{\mathbb{C}}^{(j)}$ is a parabolic sub algebra of $\mathfrak{g}_{\mathbb{C}}$ with a Levi part $l = \mathfrak{g}_{\mathbb{C}}^{(0)}$ and nilradical $u = \bigoplus_{j \epsilon \mathbb{N}^*} \mathfrak{g}_{\mathbb{C}}^{(j)}$. Call $\mathfrak{q}$ the Jacobson-Morosov parabolic subalgebra of X relative to the triple (H, X, Y). Furthermore, any two triples containing X are conjugate under $G_{\mathbb{C}}$. A nilpotent element X is **distinguished** if the only Levi subalgebra containing X is $\mathfrak{g}_{\mathbb{C}}$ itself. Hence, X is distinguished in any minimal Levi subalgebra containing it. It turns out that any two minimal Levi subalgebras containing X are conjugate under $G_{\mathbb{C}}^X$.

A parabolic subalgebra with Levi decomposition $p = t \oplus v$ is said to be distinguished if

$$\dim t = \dim \tfrac{v}{[v,v]}$$

Retaining the above notation we have:

Theorem (Bala, Carter). *There is a one-to-one correspondence between nilpotent orbits of* $\mathfrak{g}_{\mathbb{C}}$ *and G-conjugacy classes of pairs* (l, p_l)*, where l is a Levi subalgebra of* $\mathfrak{g}_{\mathbb{C}}$ *and* p_l *is a distinguished parabolic subalgebra of the semisimple algebra* $[l, l]$*.*

What really makes the correspondence possible is the fact that the orbit of a distinguished X is uniquely determined by the Jacobson-Morozov $\mathfrak{q}$ attached to X or any of its representatives. It is the Richarson orbit attached to $\mathfrak{q}$. This fails for arbitrary orbit.

There is a bijection between unipotent orbits of semisimple algebraic groups and nilpotent orbits of their Lie algebra. Hence, the above classification is extended to semisimple algebraic groups.

2. An extension of the Bala Carter theory

We present a new classification of nilpotent orbits of real reductive Lie algebras under the action of their adjoint group. Let $\mathfrak{g}$ be a real reductive Lie algebra with adjoint group G and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Also let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. Finally, let θ be a Cartan involution of $\mathfrak{g}$ and σ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with regard to $\mathfrak{g}$. Then, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are obtained by complexifying $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Denote by $K_{\mathbb{C}}$ the connected subgroup of the adjoint group $G_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, with Lie algebra $\mathfrak{k}_{\mathbb{C}}$. We prove that the orbits $K_{\mathbb{C}}.e$ are in one-to-one correspondence with the triples of the form $(l, q_l, \mathfrak{w})$, where e is a non zero nilpotent in $\mathfrak{p}_{\mathbb{C}}$, l is a minimal (θ, σ)-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing e, q_l is a θ stable parabolic subalgebra of $[l, l]$ and $\mathfrak{w}$ is a certain $L \cap K_{\mathbb{C}}$ prehomogeneous subspace of $q_l \cap \mathfrak{p}_{\mathbb{C}}$ containing e. L is the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra l. Thus, we obtain a classification for real nilpotents G-orbits in $\mathfrak{g}$ via the Kostant-Sekiguchi correspondence. This classification generalizes the one given by P. Bala and R. Carter in 1976, for complex semisimple Lie algebras.

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