Richardson Orbits For Real Semisimple Lie Groups

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PLAN OF THE TALK

- Richardson theorem for complex Lie groups
- Connection with the Bala-Carter classification
- Kostant-Sekiguchi correspondence
- Extension of Bala-Carter to Real Groups
- Connection with the work of E. Sommers
- Richardson theorem for the real Lie groups
- Tauvel's Counter-Example
- Trapa's results (Classical groups)
- Tauvel's polarization result
- Extension of Trapa's Result to Exceptional groups
- Conclusion

- $\mathfrak{g}_{\mathbb{C}}$ a semisimple complex Lie algebra

- $G_{\mathbb{C}}$ adjoint group of $\mathfrak{g}_{\mathbb{C}}$

- $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$

- Q cncted Lie subgroup of $G_{\mathbb{C}}$ such that $\mathrm{Lie}(Q) = \mathfrak{q}$

Theorem (Richardson 1974). There exists $v \in \mathfrak{u}$ such that $Q{\cdot}v$ is an open and dense subset of $\mathfrak{u}$.

In other words the pair $(Q, \mathfrak{u})$ is a prehomogeneous space in the sense of Sato and Kimura.

Moreover there exists a unique nilpotent orbit $\mathcal{O}_{\mathfrak{g}_{\mathbb{C}}}$ of $G_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ such that :

$\mathcal{O}_{\mathfrak{g}_{\mathbb{C}}} \cap \mathfrak{u}$ is an open and dense subset of $\mathfrak{u}$

$\mathcal{O}_{\mathfrak{g}_{\mathbb{C}}} \cap \mathfrak{u}$ is single Q-orbit in $\mathfrak{u}$.

$O_{\mathfrak{g}_{\mathbb{C}}}$ is called a *Richardson* orbit.

For example the principal orbit is always Richardson and corresponds to a Borel subgroup.

Variations of Richardson's theorem have been extensively studied by Gerhard Röhrle and his collaborators.

The Bala-Carter Correspondence:

[1976]. There is a 1-1 correspondence between nilpotent orbits of $G_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ and $G_{\mathbb{C}}$-classes of pairs (l, p), where l is a Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and p a **distinguished** parabolic subalgebra of $[l, l]$.

A nilpotent $E \in \mathfrak{g}_{\mathbb{C}}$ is distinguished if the only Levi that contains it is $\mathfrak{g}_{\mathbb{C}}$.

A parabolic subalgebra is distinguished if it contains a distinguished nilpotent.

A nilpotent element is always distinguished in a minimal Levi that contains it.

The orbit $\mathcal{O}_E$ of a distinguished nilpotent element E is said to be distinguished also. And such an orbit is always even.

Jacobson-Morozov Theorem

Let E be a nilpotent element of $\mathfrak{g}_{\mathbb{C}}$ then there exists a semisimple element H and a nilpotent element F in $\mathfrak{g}_{\mathbb{C}}$ such that:

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = H$$

The triple (H, E, F) is called a standard $\mathfrak{sl}_2$-triple.

Jacobson-Morozov Parabolic Subalgebra

The standard triple (H, E, F) defines a grading:

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^i$$

where $\mathfrak{g}_{\mathbb{C}}^i = \{Z \in \mathfrak{g}_{\mathbb{C}} : [H, Z] = iZ\}$.

Define $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ with $\mathfrak{l} = \mathfrak{g}_{\mathbb{C}}^0$ and $\mathfrak{u} = \bigoplus_{i>0} \mathfrak{g}_{\mathbb{C}}^i$.

$\mathfrak{l}$ is a reductive subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{u}$ is a nilpotent vector space.

$\mathfrak{q}$ is called the Jacobson-Morozov parabolic subalgebra of E. It is uniquely determined by E for any two Jacobson-Morozov parabolic subalgebras of E are conjugate under $G_{\mathbb{C}}^E$, the centralizer of E in $G_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}}^E \subset \mathfrak{q}$ (Kostant)

Richardson Orbits : Bala-Carter Correspondence

Any distinguished orbit $\mathcal{O}_E$ is uniquely determined by the Jacobson-Morozov parabolic subalgebra $\mathfrak{q}$ of any of its representatives. It is the Richardson orbit attached to $\mathfrak{q}$.

$$\text{(Bala-Carter) } \mathcal{O}_E \text{ is distinguished} \Longleftrightarrow \dim \mathfrak{l} = \dim \tfrac{\mathfrak{u}}{[\mathfrak{u},\mathfrak{u}]}$$

Such $\mathfrak{q}$ is said to be distinguished: A Borel subalgebra is always distinguished and corresponds to the principal orbit.

Hence in one direction one can define an injective map from distinguished orbits to conjugacy classes of distinguished parabolic subalgebras:

$$\mathcal{O}_E \longrightarrow [\mathfrak{q}] \text{of any Jacobson-Morozov parabolic } \mathfrak{q} \text{ of } E.$$

The classification is completed by showing that any distinguished parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus u$ is the Jacobson-Morozov parabolic subalgebra of a distinguished nilpotent E (relative to a suitable standard triple).

The Kostant-Sekiguchi Correspondence

In 1998, we extended the Bala-Carter correspondence to real reductive Lie groups. The Kostant-Sekiguchi correspondence played a key rôle in the project. Some definitions:

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: Cartan decomposition
- Θ : Cartan involution for $\mathfrak{g}$
- G adjoint group of $\mathfrak{g}$
- $K \subseteq G$:maxl compact $Lie(K) = \mathfrak{k}$
- $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$: Complexification
- σ conjugation of $\mathfrak{g}_{\mathbb{C}}$ wrt $\mathfrak{g}$
- $K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$: $Lie(K_{\mathbb{C}}) = \mathfrak{k}_{\mathbb{C}}$
- e nilpotent in $\mathfrak{p}_{\mathbb{C}}$

$K_{\mathbb{C}}$ preserves $\mathfrak{p}_{\mathbb{C}}$.

- $\{x, e, f\}$: normal $\mathfrak{sl}_2$-triple: $x \in \mathfrak{k}_{\mathbb{C}}$: $e, f \in \mathfrak{p}_{\mathbb{C}}$

Moreover e is $K_{\mathbb{C}}$-conjugate to a nilpotent e' inside of a normal triple (x', e', f') with $\sigma(e') = f'$ [**Sekiguchi**]. The triple (x', e', f') will be called a $Kostant - Sekiguchi$ or KS-triple .

Every nilpotent E' in $\mathfrak{g}$ is G-conjugate to the element E of a triple (H, E, F) in $\mathfrak{g}$ with the property that $\theta(H) = -H$ and $\theta(E) = -F$ [**Sekiguchi**]. Such a triple will be called a KS-triple also.

Define a map c from the set of KS-triples of $\mathfrak{g}$ to the set of KS- triples of $\mathfrak{g}_{\mathbb{C}}$ as follows:

$$x = c(H) = i(E - F)$$

$$e = c(E) = \tfrac{1}{2}(H - i(E + F))$$

$$f = c(F) = \tfrac{1}{2}(H + i(E + F))$$

The triple (x, e, f) is called the Cayley transform of (H, E, F). It is easy to verify that the triple (x, e, f) is a KS-triple and that $x \in i\mathfrak{k}$.

The Kostant-Sekiguchi correspondence gives a one to one map between the set of G-conjugacy classes of nilpotents in $\mathfrak{g}$ and the $K_{\mathbb{C}}$-conjugacy classes of nilpotents in $\mathfrak{p}_{\mathbb{C}}$. This correspondence sends the zero orbit to the zero orbit and the orbit through the nilpositive element of a KS-triple to the one through the nilpositive element of its Cayley transform. [**Sekiguchi**]

The correspondence was also proved independently by D. Djoković (published) and D. R. King (Classical groups) (unpublished manuscript)

Related orbits are diffeomorphic [**Vergne,1995**]

Classification of Real Nilpotent Orbits

♦ $q = l \oplus u$: Θ-stable parabolic in $\mathfrak{g}_{\mathbb{C}}$

A nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$ is **noticed** if the only (Θ, σ)-stable Levi containing e is $\mathfrak{g}_{\mathbb{C}}$. And any $e \in \mathfrak{p}_{\mathbb{C}}$ is noticed in any minimal (Θ, σ)-stable Levi containing it.

Criterion:

Lemma 1 (N)**.** *Retain the above notations. Then a nilpotent element e in $\mathfrak{p}_{\mathbb{C}}$ is noticed if and only if* $\dim \mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.

Theorem [N 1997]. For any triple $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w})$, where $\mathfrak{w} \subseteq u \cap \mathfrak{p}_{\mathbb{C}}$ is a certain $L \cap K_{\mathbb{C}}$- module with a dense orbit, there exists a $\mathfrak{sl}_2$ normal triple (x, e, f) such that e is noticed, $\mathfrak{q}$ is the Jacobson-Morozov for (x, e, f) and $\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^2 \cap \mathfrak{p}_{\mathbb{C}}$.

(Definition of $\mathfrak{w}$ is technical refer to paper)

Classification Theorem [N 1997].

There is a 1-1 correspondence between $K_{\mathbb{C}}$- nilpotent orbits on $\mathfrak{p}_{\mathbb{C}}$ and $K_{\mathbb{C}}$-classes of certain triples $(l, q_l, \mathfrak{w}_l)$

For details see:

(N.) *Nilpotent Orbits and Theta-Stable Parabolic Subalgebras* AMS Representation Theory Volume 2 Pages 1-32 (1998).

What Failed?

1. **Distinguished** nilpotent orbits are even. There are non-even **noticed** nilpotent orbits.

2. Distinguished parabolic subalgebras are defined by: $\dim \mathfrak{l} = \dim \frac{\mathfrak{u}}{[\mathfrak{u},\mathfrak{u}]}$

The obvious extension $\dim \mathfrak{l} \cap \mathfrak{k}_{\mathbb{C}} = \dim \frac{\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}}{[\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}},\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}]}$ does not work. There are noticed nilpotent elements whose theta-stable Jacobson-Morozov parabolic subalgebras do not satisfy the above formula.
We shall give the following example.

Let $\mathfrak{g}$ be $\mathfrak{sl}(7,\mathbb{R})$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(7,\mathbb{C})$, $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(7,\mathbb{C})$, and $\mathfrak{p}_{\mathbb{C}}$ is the space of 7×7 complex symmetric matrices. The Cartan involution θ is defined as $\theta(X) = -X^T$ for $X \in \mathfrak{g}$. Let

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$E = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Choose $F = E^T$. Then $\theta(H) = -H$, $\theta(E) = -F$. Hence (H, E, F) is a KS-triple, and $x = i(E - F)$ is in $\mathfrak{k}_{\mathbb{C}}$. In fact under the Kostant-Sekiguchi map (H, E, F) corresponds to a normal triple (x, e, f), with $e = \frac{1}{2}(E + F + iH)$ and $f = \frac{1}{2}(E + F - iH)$.

A simple computation shows that $\dim \mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}} = 5$. Hence the triple (x, e, f) is noticed.

From the theory of the classification of the real nilpotent orbits of $\mathfrak{sl}_n$ and by Sekiguchi

$$G_{\mathbb{C}}.e = G_{\mathbb{C}}.E,$$

and the triple (x, e, f) can be associated to the partition $[4, 2, 1]$ of 7 and the corresponding weighted Dynkin diagram is

$$2 \quad 0 \quad 1 \quad 1 \quad 0 \quad 2$$

$$o\_\_\_\_ \ \mathrm{o}\_\_\_\_ \ \mathrm{o} \ \_\_\_ \ \mathrm{o}\_\_\_\_ \ \mathrm{o}\_\_\_\_ \ \mathrm{o}$$

One sees that

$$\dim \mathfrak{g}_{\mathbb{C}}^{(1)} = 4, \, \dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}} = 2.$$

But

$$\dim[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}] = 3$$

So $\dim l \cap \mathfrak{k}_{\mathbb{C}} = 5$

while

$$\dim \frac{\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}}{[\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}]} = \dim(\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}} \oplus \frac{\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}}{[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}]}) = 4$$

It turns out the formula works for Jacobson-Morozov parabolic subalgebras attached to even noticed nilpotent orbits.

Theorem 1 (N). *Let e be an even nilpotent element of $\mathfrak{p}_{\mathbb{C}}$. Let $\mathfrak{q}$ be a θ-stable Jacobson-Morozov parabolic subalgebra of e relative to a triple (x, e, f) defined as above. Then $\mathfrak{q} = l \oplus u$ and dim $l \cap \mathfrak{k}_{\mathbb{C}}$ $=$ dim $\frac{u\cap\mathfrak{p}_{\mathbb{C}}}{[u\cap\mathfrak{k}_{\mathbb{C}}, u\cap\mathfrak{p}_{\mathbb{C}}]}$ if and only if e is noticed.*

Proof. We may assume that $\mathfrak{q}$ is defined as above. Let $u' = \bigoplus_{i\geq 4} \mathfrak{g}_{\mathbb{C}}^{(i)}$. Then

$$\dim l \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}},$$

while

$$\dim \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}} = \dim u \cap \mathfrak{p}_{\mathbb{C}} \text{ - } \dim u' \cap \mathfrak{p}_{\mathbb{C}}.$$

By definition we have

$$[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \subset u' \cap \mathfrak{p}_{\mathbb{C}}.$$

By the representation theory of $\mathfrak{sl}_2$,

if $z \in \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$ and $i \geq 4$ then
$z = [e, z']$ for some z' in $\mathfrak{g}_{\mathbb{C}}^{(i-2)} \cap \mathfrak{k}_{\mathbb{C}} \subset u \cap \mathfrak{k}_{\mathbb{C}}$.

Hence

$$u' \cap \mathfrak{p}_{\mathbb{C}} \subset [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}].$$

The conclusion follows at once.

For exceptional Lie algebras one obtains the following:

Proposition 1 (N)**.** *Let* $\mathfrak{g}_{\mathbb{C}}$ *be an exceptional simple complex Lie algebra. Let* $\mathfrak{q} = l \oplus u$ *be a* θ*-stable Jacobson-Morozov parabolic subalgebra of* e *relative to a normal triple* (x, e, f) *defined as above. If* e *is noticed then dim* $l \cap \mathfrak{k}_{\mathbb{C}} = \dim \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.

Proof. (Use Djoković's data)

3. The prehomogeneous space $\mathfrak{w}$ helps deal with the following cases:

$$K_{\mathbb{C}} \cdot e \longleftrightarrow (l, q_l, \mathfrak{w}_l^1)$$

$$K_{\mathbb{C}} \cdot e' \longleftrightarrow (l, q_l, \mathfrak{w}_l^2)$$

A connection with the work of E. Sommers

Let $G_{\mathbb{C}}^e$ centralizer of e in $G_{\mathbb{C}}$. $G_{\mathbb{C}}^e$ is not connected in general. Often it is required to compute the finite group

$$A(e) = \frac{G_{\mathbb{C}}^e}{(G_{\mathbb{C}}^e)_\circ}$$

where $(G_{\mathbb{C}}^e)_\circ$ is the identity component.

E. Sommers (1997) gave a unified description of the conjugacy classes of $A(e)$.

A pseudo-Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}_{\mathbb{C}}$ is defined to be the centralizer in $\mathfrak{g}_{\mathbb{C}}$ of a semisimple element z of $G_{\mathbb{C}}$.

A nilpotent element e of $\mathfrak{g}_{\mathbb{C}}$ is called **distinguished** if the conditions $x \in \mathfrak{g}_{\mathbb{C}}$ semisimple and $[x, e] = 0$ imply that x is in the center of $\mathfrak{g}_{\mathbb{C}}$.

Theorem 2. *(Sommers). There is a bijection Φ between $G_{\mathbb{C}}$ conjugacy classes of pairs (l, e), where l is a pseudo-Levi subalgebra and e is a distinguished nilpotent in l, and $G_{\mathbb{C}}$ conjugacy classes of pair $(e, \bar{c})$, where e is a nilpotent element in $\mathfrak{g}_{\mathbb{C}}$ and $\bar{c}$ is a conjugacy class of $A(e)$.*

In fact $(e, 1)$ always goes to the Bala-Carter Levi l.

[Sommers] *Nilpotent Orbits and the Affine Flag Manifold.* PhD. thesis MIT Sept 1997.

We have extended Sommers's result to reductive real Lie groups.

Let e be a nilpotent element in $\mathfrak{p}_{\mathbb{C}}$ and $A_k(e) = \frac{K^e_{\mathbb{C}}}{(K^e_{\mathbb{C}})_\circ}$. Give a unified description of the elements of $A_k(e)$.

A pseudo-Levi subalgebra l of $\mathfrak{g}_{\mathbb{C}}$ is said to be **elliptic** if it is the centralizer in $\mathfrak{g}_{\mathbb{C}}$ of an elliptic element of $K_{\mathbb{C}}$.

A nilpotent element e of $l \cap \mathfrak{p}_{\mathbb{C}}$ is called **noticed** if the conditions $z \in l \cap \mathfrak{k}_{\mathbb{C}}$ semisimple and $[z, e] = 0$ imply that z is in the center of l.

Theorem 3. *(King, N). There is a one to one correspondence between $K_{\mathbb{C}}$-conjugacy classes of pairs $(e, \bar{z})$, where e is a nilpotent in $\mathfrak{p}_{\mathbb{C}}$, $\bar{z} \in A_k(e)$ and $K_{\mathbb{C}}$-conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w})$ where l is an elliptic pseudo-Levi subalgebra in which e is noticed, $\mathfrak{q}_l$ is a noticed parabolic of l for e, and $\mathfrak{w}$ is a certain prehomogeneous space.*

In fact $(e, 1)$ always goes to this (Θ, σ)-stable Levi l in our extension of the Bala-Carter classification .

[**King &N**] *Component Groups of Centralizers of Nilpotents in Complex Symmetric Spaces* Journal of Algebra, **(232)** 94-125, 2000.

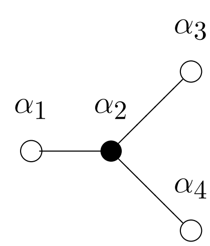
Richardson's Theorem fails for symmetric spaces

Maintaining the above notations, a straightforward extension of the Richardson's Theorem would be:

Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with Lie group $Q \subseteq G_{\mathbb{C}}$. Does $Q \cap K_{\mathbb{C}}$ admit an open dense orbit in $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$?

This is not true in general. Here is a counter-example due to Patrice Tauvel.

Let $\mathfrak{g} = \mathfrak{so}(4,4)$. Then we can represent $\mathfrak{g}$ by the following Vogan diagram:



where α_2 is the unique non-compact simple root and

$$\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(4)$$

$\mathfrak{g}$ is of inner-type.

$$\mathfrak{k}_{\mathbb{C}} = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2} \oplus \mathbb{C}H_{\alpha_3} \oplus \mathbb{C}H_{\alpha_4} \oplus \mathbb{C}X_{\pm\alpha_1} \oplus \mathbb{C}X_{\pm\alpha_3} \oplus \mathbb{C}X_{\pm\alpha_4} \oplus \mathbb{C}X_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}$$

$$\mathfrak{p}_{\mathbb{C}} = \mathbb{C}X_{\pm\alpha_2} \oplus \mathbb{C}X_{\pm(\alpha_1+\alpha_2)} \oplus \mathbb{C}X_{\pm(\alpha_2+\alpha_3)} \oplus \mathbb{C}X_{\pm(\alpha_2+\alpha_4)} \oplus \mathbb{C}X_{\pm(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}X_{\pm(\alpha_1+\alpha_2+\alpha_4)} \oplus \mathbb{C}X_{\pm(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}X_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$$

Let $\mathfrak{q}$ be the Borel subalgebra defined by the above system of simple roots then clearly $\mathfrak{q}$ is θ-stable for

$$\mathfrak{q} = (\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}) \oplus \mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}})$$

$$\dim(\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}) = \dim(\mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}}) = 8$$

$$\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}}$$

Let $z \in \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$. Then:

$$[z, X_{(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}] = 0$$

Hence $\dim[z, \mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}] \leq 7 < \dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$.

Consequently $[z, \mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}] \neq \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$.

Two good cases

Let (x, e, f) to be a normal $\mathfrak{sl}_2$-triple with $x \in i\mathfrak{k}$ e and $f \in \mathfrak{p}_{\mathbb{C}}$. Let $\mathfrak{q}$ be the θ-stable Jacobson-Morozov parabolic subalgebra of e relative to the triple (x, e, f).

Retain the above notations. Let Q be the connected subgroups of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$.

Proposition 2. *$Q \cap K_{\mathbb{C}}.e$ is a dense open subset of $\bigoplus_{i\geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$. Moreover if e is even, that is $\mathfrak{g}_{\mathbb{C}}^{(i)} = 0$ for i odd, then $\overline{Q \cap K_{\mathbb{C}} \cdot e} = u \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. The map $z \to Ad_z(e)$ is a morphism from $Q \cap K_{\mathbb{C}}$ to $\bigoplus_{i\geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$ and its differential is the map:

$$-\operatorname{ad}_e : \mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}} \to \bigoplus_{i\geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}.$$

This map is surjective. Thus the given morphism is dominant and separable. Since the image of such a map is open in its closure, the $Q \cap K_{\mathbb{C}}$-orbit of e is a non-empty open dense subset of $\bigoplus_{i\geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$.

If e is even then $\bigoplus_{i\geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}} = u \cap \mathfrak{p}_{\mathbb{C}}$. Hence, $\overline{Q \cap K_{\mathbb{C}} \cdot e} = u \cap \mathfrak{p}_{\mathbb{C}}$.

$\square$

Borel-de Siebenthal parabolic subalgebras

A complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is said to be *graded* if $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k=-\infty}^{\infty} \mathfrak{g}_{\mathbb{C}}^k$ where $\mathfrak{g}_{\mathbb{C}}^k$ is a vector subspace of $\mathfrak{g}_{\mathbb{C}}$ and $[\mathfrak{g}_{\mathbb{C}}^i, \mathfrak{g}_{\mathbb{C}}^j] = \mathfrak{g}_{\mathbb{C}}^{i+j}$ for all integers i and j.
We shall need the following theorem of Vinberg:

Theorem 4. *Let $G_{\mathbb{C}}$ be a complex semisimple Lie group with graded Lie algebra $\mathfrak{g}_{\mathbb{C}} = \bigoplus_k \mathfrak{g}_{\mathbb{C}}^k$, and let $G_{\mathbb{C}}^0$ be the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}^0$. Then the adjoint action of $G_{\mathbb{C}}^0$ on $\mathfrak{g}_{\mathbb{C}}^1$ has only finitely many orbits. Hence one of them must be open.*

$\square$

A proof of the uniqueness and denseness of such an open orbit is found in Knapp's *Lie Groups Beyond an Introduction* Proposition 10.1.

Let $\mathfrak{g}$ be of inner-type, that is $\operatorname{rank}(\mathfrak{k}) = \operatorname{rank}(\mathfrak{g})$ and Δ a Vogan set of simple roots of $\mathfrak{g}_{\mathbb{C}}$. Then Δ can be partitioned into two disjoint sets: $\Delta_{\mathfrak{k}_{\mathbb{C}}}$ the set of compact roots and $\Delta_{\mathfrak{p}_{\mathbb{C}}}$ the set of imaginary non-compact roots. Let α_p be a non-compact imaginary simple root such that if $\beta = \sum_{k=1}^{l} c_k \alpha_k$ is a positive root then $0 \leq c_p \leq 2$. Thus,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{-2} \oplus \mathfrak{g}_{\mathbb{C}}^{-1} \oplus \mathfrak{g}_{\mathbb{C}}^{0} \oplus \mathfrak{g}_{\mathbb{C}}^{1} \oplus \mathfrak{g}_{\mathbb{C}}^{2}$$

is a grading of $\mathfrak{g}_{\mathbb{C}}$ where $\mathfrak{g}_{\mathbb{C}}^i$ is the sum of the roots spaces for roots whose coefficient of α_p is i in an expansion in terms of simple roots in Δ. Define $l = \mathfrak{g}_{\mathbb{C}}^0$ and $u = \mathfrak{g}_{\mathbb{C}}^1 \oplus \mathfrak{g}_{\mathbb{C}}^2$. Then $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a maximal parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and is called a Borel-de Siebenthal parabolic subalgebra. Furthermore, $\mathfrak{p}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^1 \oplus \mathfrak{g}_{\mathbb{C}}^{-1}$. Denote by Q the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$. Then $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^1$ is a $Q \cap K_{\mathbb{C}}$-module under the adjoint action which we shall identify with its diffrential $ad : \mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}} \to \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$

Proposition 3. *$Q \cap K_{\mathbb{C}}$ has a unique open dense orbit in $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$.*

*P*roof. Observe that $\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^0 \oplus \mathfrak{g}_{\mathbb{C}}^2$ and that $\mathfrak{g}_{\mathbb{C}}^2$ acts trivially on $\mathfrak{g}_{\mathbb{C}}^1$. Therefore, the adjoint action of $\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}^1$ is equivalent to that of $\mathfrak{g}_{\mathbb{C}}^0$ on $\mathfrak{g}_{\mathbb{C}}^1$. The proposition follows from Vinberg's theorem.

□

Variations on a theorem of Peter Trapa

We say that a nilpotent orbit $\mathcal{O}_k$ of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ is a *Richardson* orbit if there exists a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ such that $\mathcal{O}_k$ is the unique dense orbit admitted by the saturation of $K_{\mathbb{C}}$ on $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$. That is $\mathcal{O}_k \cap (\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$ is open and dense in $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$. In the case where $G_{\mathbb{C}}$ is a classical complex Lie group Peter Trapa proves the following theorem:

Theorem 5. *Fix a special nilpotent orbit $\mathcal{O}$ of $G_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$. Then there exists a real form G such that some irreducible component of $\mathcal{O} \cap \mathfrak{p}_{\mathbb{C}}$ is a Richardson orbit of $K_{\mathbb{C}}$ on the nilpotent cone of $\mathfrak{p}_{\mathbb{C}}$.*

Source: (P. Trapa) *Richardson Orbit for Real Classical Groups* To appear in Journal of Algebra.

We shall see that the above theorem does not extend to exceptional groups. It fails for the minimal orbits of E_7 and E_8.

Definition 1. A nilpotent orbit of the real group G on $\mathfrak{g}$ is a Richardson orbit if its image under the Kostant-Sekiguchi correspondence is a Richardson orbit as defined above.

More of Trapa's results

Using the explicit computation of Richardson orbits in the classical cases, Trapa establishes the following:

Theorem 6. *For the classical groups, the annihilator of any module of the form $A_{\mathfrak{q}}(\lambda)$ is explicitly computable.*

The definition of $A_{\mathfrak{q}}(\lambda)$ is found in Knapp and Vogan book "Cohomological Induction and Unitary Representations". ($A_{\mathfrak{q}}$ is related to a real form.)

Using the preceding theorem Trapa gives new examples of simple highest weight modules with irreducible associated varieties via the following:

Theorem 7. *Fix $G_{\mathbb{C}}$ complex semisimple (not necessarily classical). Suppose I is the annihilator of an $A_{\mathfrak{q}}$ module for some real form G of $G_{\mathbb{C}}$. If $Ann_{\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})}(L(w_1)) = I$, then $AV(L(w))$ is irreducible , i.e. is the closure of a unique orbital variety for $\mathfrak{g}_{\mathbb{C}}$. If $G_{\mathbb{C}}$ is classical, this orbital variety is effectively computable*

Tauvel's result

Definition 2. $\mathfrak{q}$ is a *polarization* of $\mathfrak{g}_{\mathbb{C}}$ at e if

$$2\dim\mathfrak{q} = \dim\mathfrak{g}_{\mathbb{C}}^{e} + \dim\mathfrak{g}_{\mathbb{C}} \quad \text{and} \quad B(e, [\mathfrak{q}, \mathfrak{q}]) = 0$$

where $\mathfrak{g}_{\mathbb{C}}^{e}$ is the centralizer of e in $\mathfrak{g}_{\mathbb{C}}$ and B is the Killing form of $\mathfrak{g}_{\mathbb{C}}$.

The next proposition could be seen as a version of Richardson's theorem for complex symmetric spaces.

Proposition 4. *(P. Tauvel). Maintaining the above notations, suppose that there exists z in $\mathfrak{p}_{\mathbb{C}}$ such that $\mathfrak{q}$ is a polarization of $\mathfrak{g}_{\mathbb{C}}$ at z. Then*

i. There exists a unique $K_{\mathbb{C}}$-nilpotent orbit $\mathcal{O}_k$ in $\mathfrak{p}_{\mathbb{C}}$ such that $\mathcal{S} = \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} \cap \mathcal{O}_k$ is open and dense in $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$.

ii. $\mathcal{S}$ is a $Q \cap K_{\mathbb{C}}$-orbit.

iii. if $x \in \mathcal{S}$ then $[x, \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{q}] = \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$, $[x, \mathfrak{p}_{\mathbb{C}} \cap \mathfrak{q}] = \mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{q}$ is a polarization of $\mathfrak{g}_{\mathbb{C}}$ at x.

Source: (P. Tauvel), *Quelques résultats sur les algèbres de Lie symétriques*, Bull. Sci. math. **125** 8 (2001), 641-665.

If $e \in \mathfrak{p}_{\mathbb{C}}$ and $G_{\mathbb{C}} \cdot e$ is an even orbit then the Jacobson-Morozov parabolic subalgebra of e is a polarization of $\mathfrak{g}_{\mathbb{C}}$ at e.

This is due to the fact that :

$$2 \dim \mathfrak{q} = \dim \mathfrak{g}_{\mathbb{C}} + \dim \mathfrak{g}_{\mathbb{C}}^0 \, , \dim \mathfrak{g}_{\mathbb{C}}^e = \dim \mathfrak{g}_{\mathbb{C}}^0 + \dim \mathfrak{g}_{\mathbb{C}}^1$$

$$\text{with } B(e, \mathfrak{g}_{\mathbb{C}}^i) = \{0\} \text{ if } i \neq -2.$$

Remember Trapa's Theorem:

Let $\mathfrak{g}_{\mathbb{C}}$ be a classical semisimple complex Lie algebra. Fix a special nilpotent orbit $\mathcal{O}$ of $G_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$. Then there exists a real form G such that some irreducible component of $\mathcal{O} \cap \mathfrak{p}_{\mathbb{C}}$ is a Richardson orbit of $K_{\mathbb{C}}$ on the nilpotent cone of $\mathfrak{p}_{\mathbb{C}}$.

We shall now try to extend this result to the exceptional cases. Trapa's proof used the fact that in the classical case nilpotent orbits are parametrized by signed Young tableaux. This is not available for exceptional groups. However, labelled Dynkin Diagrams from the work of Dynkin and Kostant, and information from the work of Bala, Carter and Djoković is used instead.

The Exceptional Cases

From the previous remarks we need to consider only non-even special orbits. Since there are none in G_2 we conclude that Trapa's result extends to G_2.

Definition 3. An orbit $\mathcal{O}$ of $G_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ is *polarizable* if for some irreducible component of $K_{\mathbb{C}} \cdot e$ of $\mathcal{O} \cap \mathfrak{p}_{\mathbb{C}}$ there exists a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{q}$ is a polarization of $\mathfrak{g}_{\mathbb{C}}$ at e.

Strategy

1. Find all special polarizable non-even nilpotent orbits of the other complex exceptional simple Lie groups. In this case, Tauvel's result implies Trapa's result.

2. It turns out that the non-polarizable special nilpotent orbits have low dimension. Let $\mathcal{O}$ be such an orbit, using information from Djoković's classification we were able either to find an appropriate $\mathfrak{q}$ or to show that for all $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, we can find $e \in \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$ such that $\dim K_{\mathbb{C}} \cdot e > \dim \mathcal{O} \cap \mathfrak{p}_{\mathbb{C}}$.

ALGORITHMIC SCHEME

Input:

1. Type of simple Lie algebra $\mathfrak{g}$ (inner-type)
2. A Θ-stable system of roots Δ^1 (Vogan)

Computation:

3. n = # of non equivalent Θ-stable system of roots $\{n = |\frac{W(G_{\mathbb{C}})}{W(K_{\mathbb{C}})}|\}$
4. Create_list $L = \{\Delta^1 \dots \Delta^n\}$ $\{\Delta_i = S_\beta(\Delta_{i-1})\}$ β's non-compact imaginary roots.
5. Initialize storage $\mathfrak{Q}$
6. **For** each $\Delta^i \in L$ **do**
   For each subset $\mathfrak{S} \subset \Delta^i$ **do**
   Create the parabolic $\mathfrak{q}$ associated with $\mathfrak{S}$
   If $\mathfrak{q} \notin \mathfrak{Q}$ **then** Insert $(\mathfrak{Q}, \mathfrak{q})$

Output:

7. $\mathfrak{Q}$ consists of distinct $\mathfrak{q}$'s.

Source: (N.) *Computing theta-stable parabolic subalgebras using LiE.* Lectures Notes in Computer Science, Springer-Verlag Volume 3039 (335-342) 2004.

It contains a complete description, implementation details in LiE (pidgin), proof of correctness and complexity.

What happens when $\mathfrak{g}$ is not of inner-type?

The algorithm is essentially the same. But the implementation will be more complex.

We only had to deal with the real form EIV of E_6 for the minimal orbit case. The computation is quite tractable for there is only one class of roots system of interest since that real form does not have any noncompact imaginary roots. It turns out that in this case one can find a parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ such that

$$\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} = \mathbb{C}(X_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6} - X_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6})$$

Results

Trapa's Theorem extends to G_2, F_4 and E_6. It fails for the minimal orbit of E_7 and for several low dimensional orbits in E_8.

Theorem 8. *(N.) Maintaining our previous notations, let $\mathfrak{g}_{\mathbb{C}}$ be a simple complex Lie algebra other than E_8 and fix a non-minimal special nilpotent orbit $\mathcal{O}$ of $G_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$. Then there exists a real form G such that some irreducible component of $\mathcal{O} \cap \mathfrak{p}_{\mathbb{C}}$ is a Richardson orbit of $K_{\mathbb{C}}$ on the nilpotent cone of $\mathfrak{p}_{\mathbb{C}}$.*

Source: (N.) *Some remarks on Richardson Orbits in Complex Symmetric spaces.* Preprint (2004)

Conlusion

The last result has the flavor of a corresponding result for *Admissible nilpotent orbits*. Such orbits were classified by T. Ohta (1990 complete), and Schwartz (1987, partial) for classical groups. Using some facts from Ohta's paper and Djoković's classification we classified the admissible orbits for exceptional real Lie groups.

Sources:

[N] *Classification of Admissible Nilpotent Orbits in Simple Exceptional Real Lie Algebras of Inner type.* AMS Journal of Representation Theory **5** 2001 (455-493)

[N] *Classification of Admissible Nilpotent Orbits in the Simple Real Lie Algebras* $E_{6(6)}$ *and* $E_{6(-26)}$. AMS Journal of Representation Theory **5** 2001 (494-502)

If G is a split real form of a classical $G_{\mathbb{C}}$ then all real orbits that come from a special (Lusztig's sense) complex orbits are admissible (Nevins). However this fails for two orbits in $E_{8(8)}$ and for one orbit in $E_{7(7)}$. Although it is true that at least one of such real orbits will be admissible.

More conceptual arguments for these problems might be very enlightening.