The Orbit Method: An insight from Physics

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Outline of the Talk

- Short Biography
- Natural and Artistic Symmetry
- Mathematical Symmetry
- Group and Group Representation
- Nilpotent Orbits
- The Orbit Method

Birthplace: Cayes Haiti

Beautiful beaches

Some interesting facts

• In 1815 the South American liberator Simón Bolívar visited the port to accept Haitian arms and a contingent of troops to aid him in his fight against Spain.

• The town was badly damaged by fire in 1908 and by hurricane in 1954.

• Les Cayes was the scene of a massacre in December 1929 when U.S. marines killed a dozen peasants protesting poor economic conditions under U.S. occupation.

• Les Cayes is Haiti's leading southern port, exporting sugar, coffee, bananas, cotton, timber, dyewood, and hides. Historic landmarks include an arsenal and several forts dating from buccaneer times.

Catholic Elementary and Middle School. But I am and always have been a non practicing Episcopalian.

Classical French education late 19th / early 20th century

Classics, Western Civilization (France), Religion both Ancient and New testaments, Mathematics, Physics, Organic Chemistry, Greek, Latin, English, Spanish.

Began to develop interest in pure Mathematics in Middle School.

Lycée Philipe Guerrier Cayes - College Roger Anglade Port-au-Prince

Section Mathematics/Physics

First two years of High-School [Lycée Philipe Guerrier Cayes]

Expelled from the lycée after a confrontation with the " Math Teacher!"

Moved to Port-au-Prince and took the first Baccalauréat exam.

Enrolled to the College Roger Anglade and took the second Baccalauréat exam.

Done with high school but did not go to the University.

Life after High School

Took a year off (exhausted) wanted to be a philosopher.

Studied Electro-mechanic for three years in a technical/vocational school run by the French in a suburb of Port-au-Pince. [Centre Pilote de Formation Professionelle]

Continued to learn mathematics on my own with the help of a few Haitians trained in French Engineering schools and Universities.

Emigrated to the USA in 1982. Learned some Computer Science. Invited to start graduate school in Mathematics at Northeastern University in 1984. Received MS in 1986. Went to work as a software Engineer in Industry for 7 years. (Applied Differential Geometry : Curve and Surface Modelling)

Doctoral Work

In 1993 I came back to NU to work on a PhD. Completed in 1997 with a thesis in Lie Theory supervised by Donald R. King.

Went back to Industry 1997-1998 [Worked on the 2000 bug]

In 1998 I secured a position at UmassBoston gaining tenure in 2005.

END OF BIOGRAPHY ... QUESTIONS?

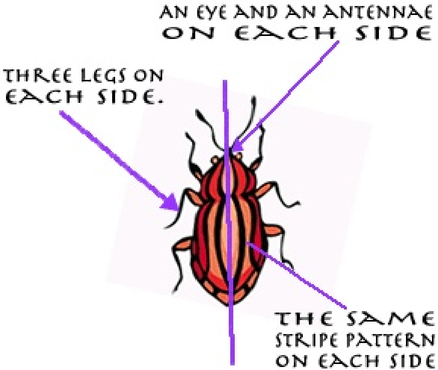
Symmetry in Nature

There is a lot of symmetry in Nature. The word comes from Greek *sym* and *metria*. It was associated to beauty by Greek and Roman philosophers:

Vitruvius in *De Architectura Libri Decem*:
" The design of a temple depends on symmetry, the principles of which must be carefully observed by the architect. They are due to proportion. Proportion is a correspondence among the measures of the members of an entire work, and the whole to a certain part selected as standard. From this result the principles of symmetry"

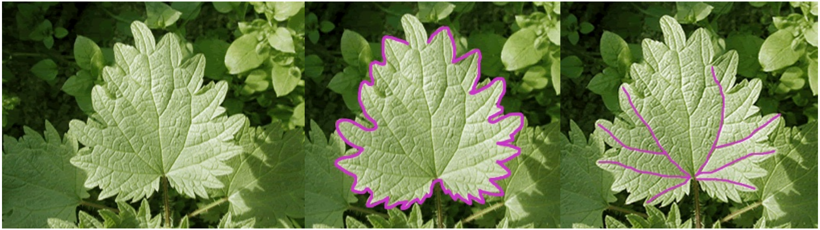
Most scientists and artists would agree that this is a description of "beauty" as it relates to their respective fields.

Symmetry in Nature



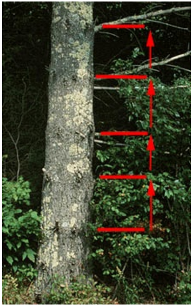
Symmetry in Nature

Same object with different types of symmetry:



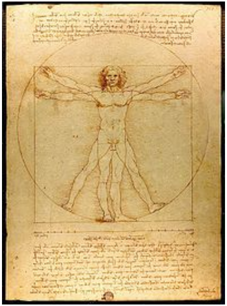
Symmetry in Nature

An Eastern White Pine has interesting symmetry on it's trunk. Each year, as the tree grows, it develops a new ring of branches. The rings move up by similiar translation vectors, but some variation occurs due to the conditions for that year.



Symmetry in Art

Leonardo da Vinci's Vitruvian Man (ca. 1487) is often used as a representation of symmetry in the human body and, by extension, the natural universe.



Symmetry in Art

The *Gelede headdress* often consists of two parts, a lower mask and an upper superstructure. The lower mask depicts a woman's face, its composure expressing the qualities of calmness, patience, and "coolness" desired in women. [Yoruba, Nigeria]

Symmetry in Mathematics

The *Birth of Venus* is a painting by Sandro Botticelli. "Most people perceive this painting as Symmetrical ..... Yet most mathematicians will tell you that the arrangements of colors and forms are not symmetric in the Mathematical sense" [Mario Livio]

Symmetry in Mathematics

Hermann Weyl " A thing is symmetrical if there is something you can do to it so that after you have finished doing it it looks the same as before."

The following stanza is symmetrical with respect to backward reading:
IS IT ODD HOW ASYMMETRICAL IS "SYMMETRY" ? "SYMMETRY" IS ASYMMETRICAL HOW ODD IT IS [Mario Livio]

Symmetry in Mathematics

• Mathematicians and scientists often used **GROUP THEORY** to study symmetry that is expressed by group transformations preserving some structure. "Evariste Galois [1811 - 1829] :
$ax^5 + bx^4 + cx^3 + dx^2 + ex + d = 0$

• Felix Klein (Das Erlanger Programm, 1872)

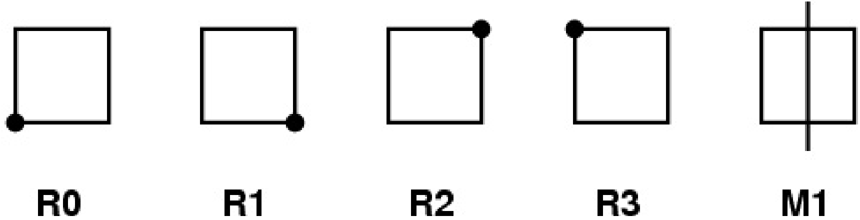
• Sophus Lie und Friedrich Engel (Theorie derTransformationsgruppen, 1888-1893)

• Elie Cartan [Géomètre Français]

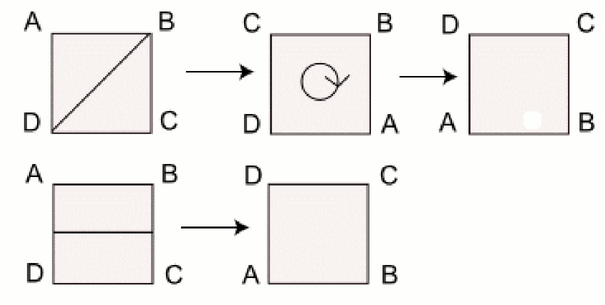
• A group $\mathfrak{G}$ is a set with an operation **like** Addition or Multiplication in the set of all real numbers.

The Group of Symmetries of the Square

The square has eight symmetries - four rotations, two mirror images, and two diagonal flips:



The Group of Symmetries of the Square



$6 \times 2 = 3 \times 4 = 12$

The Group of Symmetries of the Square

Multiplication Table

All the symmetries of the square

e identity (do nothing)
r rotate clockwise by $\pi/2$
s rotate clockwise by π
t rotate clockwise by $3\pi/2$
f flip about vertical center line
g flip about horizontal center line
h flip about diagonal from NW to SE
k flip about diagonal from NE to SW

Table for composing symmetries of the square

	e	r	s	t	f	g	h	k
e	e	r	s	t	f	g	h	k
r	r	s	t	e	k	h	f	g
s	s	t	e	r	g	f	k	h
t	t	e	r	s	h	k	g	f
f	f	h	g	k	e	s	r	t
g	g	k	f	h	s	e	t	r
h	h	g	k	f	t	r	e	s
k	k	f	h	g	r	s	t	e

Representations of a Group

A finite dimensional representation of a group is a process that associates a matrix to each element of the group.

Consider the set of permutations of 3 objects. This is a group.

$\mathfrak{S}_3 = \{\{a, b, c\}, \{a, c, b\}, \{b, a, c\}, \{b, c, a\}, \{c, a, b\}, \{c, b, a\}\}$

Representations of a Group

$$\{a, b, c\} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \{a, c, b\} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\{b, a, c\} \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \{b, c, a\} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\{c, a, b\} \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \{c, b, a\} \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

These matrices form a group $\mathfrak{G}$ under matrix Multiplication and we may think of $\mathfrak{G}$ **acting** on three dimensional vectors by permuting their components.

Parity in Quantum Mechanics

$\mathfrak{G} = \{id, r\}$ is defined by

$\times$	id	r
id	id	r
r	r	id

$\mathfrak{G}$ has exactly two irreducible representations

- Trivial representation: $id \to 1$, $r \to 1$
- Parity representation: $id \to 1$, $r \to -1$

Any other representation of $\mathfrak{G}$ must be a combination of these.
From nonrelativistic quantum mechanics in one dimension, a particle in a potential symmetric about $x = 0$ has energy eigenfunctions that are either symmetric if x is replaced by $-x$ (Trivial Representation) or antisymmetric corresponding (Parity Representation). [Howard Georgi]

The Unitary Dual

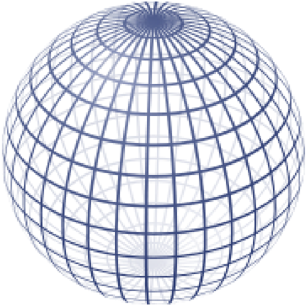
A distinguished class of representations, the irreducible unitary ones, has been shown to be the building blocks for all representations and seems to be more relevant to current applications in Physics and Number theory.

Considerable efforts have been made over the last 50 years to develop a theory capable of generating all the irreducible unitary representations, *the unitary dual*, of a given group (of Lie type). Furthermore in most of the interesting cases these representations have infinite dimension.

The answer is known in many cases. However, there are large classes of groups for which we still do not have a sufficiently well developed theory.

The orbit of a point under a group action

Let SO_3 be the real group of rotations in 3-dimensional space. If we fix a system of coordinates then the group rotates each point P in all possible ways. if P is not the origin the resulting **orbit** of P is a **sphere**. Such orbits are called *HOMOGENEOUS SPACES*. [Action is multiplication]



Geometry of nilpotent orbits

A non zero matrix A is *nilpotent* if $A^k = 0$ for some positive integer k.

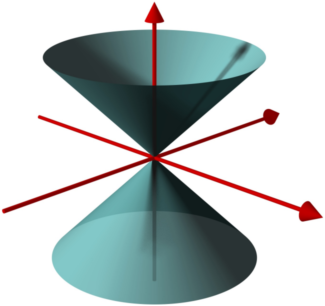
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{with} \quad k = 2$$

Let $\mathfrak{g}$ be the set of real matrices of the form $\begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}$ and $\mathfrak{G}$ be the group of matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$. Then $\mathfrak{G}$ acts on $\mathfrak{g}$ by conjugation $\mathfrak{G}\mathfrak{g}\mathfrak{G}^{-1}$. The orbit of a nilpotent matrix A, $\mathfrak{G}A\mathfrak{G}^{-1}$ is made of nilpotent matrices and is called a *nilpotent orbit*. Furthermore the number of such orbits is finite.

Description of $\mathfrak{N}$ the set of nilpotent matrices of $\mathfrak{g}$

Observe that $\begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}^{2k} = (x^2+y^2-z^2)^k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Hence $\mathfrak{N}$ is the set of points (x, y, z) such that $x^2+y^2-z^2=0$. In other words $\mathfrak{N}$ is a double sheeted-cone.



Classification of $\mathfrak{N}$ the set of nilpotent matrices of $\mathfrak{g}$

The number of nilpotent orbits corresponds to partitions of 2:

Class I: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow [2] \Longrightarrow$ Upper Cone

Class II: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow [1,1] \Longrightarrow$ zero

Class III: $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Longrightarrow [2] \Longrightarrow$ Lower Cone

The "Orbit Method" Philosophy

Let G be the group of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$.

For our purpose:

The orbit method says that irreducible representations of G "should" correspond to conjugacy classes of 2×2 *matrices: something that we teach undergraduates to parametrize, using Jordan canonical form.*

The main problem: The correspondence is not perfect. There are irreducible representation that do NOT correspond to any nilpotent orbits: [Stein, 1967 " complementary series" of $SL_2(\mathbb{R})$]

The "Orbit Method" Philosophy: Connection with Physics

"Symplectic geometry is a reasonable mathematical model for classical mechanics. The collection of all possible positions and momenta of all particles in a classical mechanical system (the phase space) is a symplectic manifold. Classical observables are functions on the phase space. An orbit that is, a homogeneous Poisson G-manifold may therefore be regarded as a classical mechanical system endowed with a group G of symmetries.

Hilbert space is a reasonable mathematical model for quantum mechanics. The collection of all (unnormalized) wave functions for a quantum mechanical system is a Hilbert space. Quantum observables are self-adjoint operators on that space. An irreducible representation may therefore be regarded as a quantum mechanical system endowed with a group of symmetries." [Vogan]

The “Orbit Method” Philosophy: Connection with Physics

“That such a quantization exists is perhaps surprising. To a quantum system one expects to be able to associate a classical system by taking Plancks constant to zero, but there is no good reason to expect that there should be a natural way of quantizing a classical system and getting a unique quantum system. Remarkably, we are able to do this for many classes of symplectic manifolds. For nilpotent groups like the Heisenberg group, that the orbit method works is a theorem [Kirillov], and this can be extended to solvable groups[Souriau, Kostant, Duflo]. What remains to be understood is what happens for reductive groups.” [Peter Woit]

Complications start with 2×2-matrices as noted above.

Geometric Quantization for Reductive Groups

Physics	**Representation Theory**
(Phase Space)	(Symplectic Homogeneous Space)
$\|$	$\|$
Classical System	"Admissible" Nilpotent Orbit
$\Downarrow$	$\Downarrow$
QM System (Hilbert Space)	Irreducible Unitary Representation of G
$\|$	$\|$
(Unitary Operators)	(Unitary Operators)

Admissible Nilpotent Orbits

Conjecture 1970's [Vogan]: These orbits are very likely to give a general parametrization of large class of irreducible unitary representations.
Until 8 years ago they were not classified!

All complex orbits are admissible. P-adic orbits[M. Nevins (1998, 2002)]

Classical Reductive Real Lie Groups [J. Schwartz (1987), T. Ohta (1991)]

Exceptional Reductive Real Lie Groups [A. Nœl (2001)]

[N-1] *Classification of Admissible Nilpotent Orbits in Simple Exceptional Lie Algebras of Inner Type.* American Mathematical Society Journal of Representation Theory, Volume 5 (455-493) 2001.

[N-2] *Classification of Admissible Nilpotent Orbits in Simple Exceptional Lie Algebras of Type* $E6(6)$ *and* $E6(-26)$. American Mathematical Society Journal of Representation Theory, Volume 5 (494-502) 2001.

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