ON THE ADMISSIBILITY OF NILPOTENT ORBITS OF SIMPLE EXCEPTIONAL REAL LIE GROUPS OF INNER TYPE

Preliminary Report

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The Orbit Method:

Questions about representations of G can be understood in terms of data attached to coadjoint orbits in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G.

Mathematical Physics: Quantization

Quantum Mechanical Model

$\downharpoonleft\upharpoonright$

Classical physical system

Representation Theory: Quantization

Unitary group representation

$\downharpoonleft\upharpoonright$

Symplectic homogeneous space

$\downdownarrows$

Covering spaces of orbits of G in $\mathfrak{g}^*$

Orbit Method works well for:

1. Nilpotent Lie Groups (Kirillov, 1962)
2. Type I Solvable Lie Groups (Auslander, Kostant 1971)
3. Compact Lie Groups

Real Reductive Lie Groups

Irreducible Unitary Representations

$\Uparrow$

Parabolic / Cohomological Induction

$\Downarrow$

Admissible Semisimple Orbits

Quantization of Nilpotent Orbits
Not well understood

Evidence of the importance of Admissible Nilpotent orbits given by Vogan and Barbasch for real Lie groups.

Definition of Admissibility for G

- G : real semisimple Lie group, $\mathfrak{g} = Lie(G)$
- $\lambda \in \mathfrak{g}^*$
- $Ad_g^*(\lambda)(E) = \lambda(Ad_{g^{-1}}(E))\ g \in G,\ E \in \mathfrak{g}$
- $ad_X^*(\lambda)(E) = -\lambda([X, E])\ X \in \mathfrak{g}$
- $G^\lambda = \{g \in G : g(\lambda) = \lambda\}$
- $\mathfrak{g}^\lambda = \{X \in \mathfrak{g} : \lambda([X, \mathfrak{g}]) = 0\}$
- ω_λ on $\mathfrak{g}/\mathfrak{g}^\lambda$, $\omega_\lambda(X + \mathfrak{g}^\lambda, Y + \mathfrak{g}^\lambda) = \lambda([X, Y])$
- Use $M(\omega_\lambda)$ and $Sp(\omega_\lambda)$ to get $\tilde{G}^\lambda$ as a double cover of G^λ.

(ρ, V) of $\tilde{G}^\lambda$ admissible if $\rho(\xi) = -1_V$ and $d\rho(E) = \sqrt{-1}\lambda(E)1_V$ **for all** $E \in \mathfrak{g}^\lambda$ **[Duflo,82]**

$$1 \longrightarrow \{1, \xi\} \longrightarrow \tilde{G}^\lambda \xrightarrow{\pi} G^\lambda \longrightarrow 1$$

λ **admissible if** $\exists$ **one admissible** ρ
$E \in \mathfrak{g}$ **admissible** $\iff$ λ_E **admissible in** $\mathfrak{g}^*$
Killing form identification

Definition of Admissibility for $\mathfrak{p}_{\mathbb{C}}$

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: Cartan decomposition
- G adjoint group of $\mathfrak{g}$
- $K \subseteq G$:maxl compact $Lie(K) = \mathfrak{k}$
- $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$: Complexification
- $K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$: $Lie(K_{\mathbb{C}}) = \mathfrak{k}_{\mathbb{C}}$
- Fix e nilpotent in $\mathfrak{p}_{\mathbb{C}}$

$K_{\mathbb{C}}$ preserves $\mathfrak{p}_{\mathbb{C}}$ and $K^e_{\mathbb{C}}$ acts on $\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}^e_{\mathbb{C}}$.

Define $\delta_e(g) = (det(g|_{\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}^e_{\mathbb{C}}}))^{-1} \qquad g \in K^e_{\mathbb{C}}$

e **admissible** $\iff \exists\chi$ **a character of** $(K^e_{\mathbb{C}})_{\circ}$ **such that** $d\chi = 1/2 d\delta_e$

It is hard to compute δ_e in general.

The two definitions are equivalent

The Kostant-Sekiguchi correspondence

- There is a 1-1 correspondence between real G-nilpotent orbits in $\mathfrak{g}$ and complex $K_{\mathbb{C}}$-nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$ [**J. Sekiguchi,1987**]

Theorem [Schwartz ,1987]

- There is a 1-1 correspondence between real G-nilpotent **admissible** orbits in $\mathfrak{g}$ and complex $K_{\mathbb{C}}$-nilpotent **admissible** orbits in $\mathfrak{p}_{\mathbb{C}}$.

Known Results on Admissibilty

1. All nilpotent orbits of complex simple Lie groups are admissible [**T. Ohta, 1991**]

2. All even nilpotent orbits are admissible. [**J. Schwartz 1987**]

3. Classification of admissible nilpotents of classical real Lie groups
[**J. Schwartz, 1987**]
[**T. Ohta, 1991**]

4. Classification of admissible nilpotents of classical p-adic groups
[**M. Nevins, 1998**]

5. All nilpotent orbits of real simply connected G_2 are admissible.
[**M. Nevins, 1998**]

Two Lemmas of Takuya Ohta

- $\{x,e,f\}$: $\mathfrak{sl}_2$-triple: $\quad x \in \mathfrak{k}_{\mathbb{C}}\ :\ e,f \in \mathfrak{p}_{\mathbb{C}}$

- $\mathfrak{k}_{\mathbb{C}}^{e} = \mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \oplus u_e$

Lemma 1: $d\delta_e|_{u_e} = 0$, $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ reductive

$\mathfrak{t}_{\mathbb{C}} \subseteq \mathfrak{k}_{\mathbb{C}}$, toral, $x \in \mathfrak{t}_{\mathbb{C}} \Rightarrow \mathfrak{t}_{\mathbb{C}}^{e}$ toral in $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$,

$$d\delta_e(t)|_{\mathfrak{t}_{\mathbb{C}}^{e}} = \sum_{i\geq 1} tr(ad(t)|_{\mathfrak{k}_{\mathbb{C}}^{i}}) - \sum_{i\geq 2} tr(ad(t)|_{\mathfrak{p}_{\mathbb{C}}^{i}})$$

$\mathfrak{k}_{\mathbb{C}}^{i}$, $\ \mathfrak{p}_{\mathbb{C}}^{i}$: i-eigenspace of ad_x on $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$

Lemma 2: $\mathfrak{t}_1$ CSA of $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$, T_1 corresponding connected subgroup of $(K_{\mathbb{C}}^{(x,e,f)})_{\circ}$.

e **admissible** $\Longleftrightarrow \exists\chi : \chi^2_{|T_1} = \delta_e$

It is not easy to compute $\mathfrak{t}_1$ from the data

$T_1 \backsimeq (\mathbb{C}^{\times})^r \quad d\delta_e : \mathbb{C}^r \to \mathbb{C}$ then e admissible iff the coeff of z_i in $d\delta_e(z_1 \ldots z_r)$ is even

New results on admissibility

- e is **noticed** $\iff \mathfrak{k}_{\mathbb{C}}^{(x,e,f)} = 0$

- Noticed nilpotents classified all nilpotents **[N, 1997]**

Theorem 1 [N,01]: Noticed $\Rightarrow$ Admissible

The admissible nilpotent orbits of all simple real exceptional Lie groups are classified via an algorithm [**N,2001**]

Theorem 2 [N,2001] If $\mathfrak{g}$ is a non-split real form of equal rank other than EII, the rank 4 form of E_6, then the nilpotent orbits of the simply connected G on $\mathfrak{g}$ are admissible.

An Example: $\mathfrak{g} = G_2$, $\mathfrak{k}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$

Orbit : vector for a short noncompact root coming from the 8 dimensional complex orbit

Triple: $(H_{2\alpha_1+\alpha_2}, X_{2\alpha_1+\alpha_2}, X_{-2\alpha_1-\alpha_2})$

Simple roots for $\mathfrak{k}_{\mathbb{C}}$: $\quad \alpha_1$, $3\alpha_1 + 2\alpha_2$

max torus : $\mathfrak{t}^1_{\mathbb{C}} = \mathbb{C}H_{\alpha 2}$

$d\delta_e(z_1 H_{\alpha_2}) = z_1$

Not admissible if G is adjoint

Assume G simply connected.

$K_{\mathbb{C}}$- fundamental weight $\lambda_1 = \alpha_1/2$

$\lambda_1(z_1 H_{\alpha_2}) = z_1/2$

$\chi = \lambda_1 \Rightarrow$ **Admissibility**

Connections with Special Orbits

A complex nilpotent special orbit is one associated to a special representation of the Weyl group via the Springer correspondence [**Lustig, 1978**]. They are one of the important tools in the classification of primitive ideals of $\mathfrak{U}(\mathfrak{g})$.

A real nilpotent orbit is special if it comes from a special complex nilpotent orbit.

Theorem 3, [**N,2001**] If $\mathfrak{g}$ is of type EIII, EVI, EVII, EIX, FI, FII, G then all special nilpotent orbits of the simply connected group are admissible.

Recently, **Monica Nevins** has computed the admissible nilpotents of the split forms of F_4, E_6, E_7 using techniques for p-adic groups.

An Interesting Case to Investigate

Conjecture: if G is a real, simply connected, split form of an exceptional Lie group then all special nilpotent orbits are admissible.

This fails for the split forms of of E_7 and E_8. Curiously it happens for 2 special orbits in EV and one in $EVIII$; all contained in a minimal algebra $A_4 \oplus A_1$.

In all cases there is an admissible real orbit coming from the same special complex orbit.

In $G_{\mathbb{C}}$ Irreducilble characters of W correspond to rational Hecke. alg. rep.
[**Curtis, Lustig, Springer**]

Only charaters associated with orbits $A_4 \oplus A_1$ in E_7 , $A_4 \oplus A_1$ and $E_6(a_1) \oplus A_1$
do not correspond to rat. Hecke alg. rep.