A LiE subroutine for Computing Prehomogeneous Spaces Associated with Real Nilpotent Orbits

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Relevant Papers of Jackson and Noël

1. **Prehomogeneous spaces associated with complex nilpotent orbits**, 43 pages, to appear in Journal of Algebra, doi:10.1016/j.jalgebra.2005.02.017.

2. **A LiE subroutine for computing prehomogeneous spaces associated with complex nilpotent orbits**, Lecture Notes in Computer Science 3516 (2005), pp. 611–618.

*****3. **A LiE subroutine for computing prehomogeneous spaces associated with real nilpotent orbits**, Lecture Notes in Computer Science 3482 (2005), pp. 512–521.

4. **Prehomogeneous spaces associated with nilpotent orbits in simple real Lie algebras $E_{6(6)}$ and $E_{6(-26)}$**, 22 pages, submitted to American Mathematical Society Mathematics of Computation Journal.

5. **Prehomogeneous spaces associated with real nilpotent orbits**, 80 pages, Preprint.

PREHOMOGENEOUS SPACES

• $\mathfrak{g}$ semisimple complex Lie algebra ($n \times n$ matrices of trace zero for example)

• G adjoint group of $\mathfrak{g}$ ($n \times n$ matrices with determinant 1 for example)

• V vector space on $\mathbb{C}$

• ρ regular representation of G on V (associates to each element of G a linear transformation in $Aut(V)$ whose entries are polynomials

(G, ρ, V) or (G, V) is called a **Prehomogenous space** if there exists a G-orbit Ω that is Zariski-open. (fill the space V) (M Sato) (New kind of zeta functions)

Examples:

1. $G = C^{*}$, $V = \mathbb{C}$, $\rho(t)x = tx$, $\Omega = \mathbb{C}^{*}$

2. $G = GL(n), V = \mathbb{C}^{n}, \rho(A)x = Ax, \Omega = \{x \neq 0\}$

GRADED LIE ALGEBRAS

A complex Lie algebra $\mathfrak{g}$ is said to be *graded* if $\mathfrak{g} = \bigoplus_{k=-\infty}^{\infty} \mathfrak{g}^k$ where $\mathfrak{g}^k$ is a vector subspace of $\mathfrak{g}$ and $[\mathfrak{g}^i, \mathfrak{g}^j] = \mathfrak{g}_{\mathbb{C}}^{i+j}$ for all integers i and j.

Theorem 1. *(Vinberg) Let G be a complex semisimple Lie group with graded Lie algebra $\mathfrak{g} = \bigoplus_i \mathfrak{g}^i$, and let G^0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}^0$. Then under the adjoint action (G_0, g^i) are prehomogeneous spaces for $i \neq 0$.*

REAL LIE GROUPS

The theory is more complicated and very much incomplete.

Some Definitions : $\mathfrak{g}$ real semisimple Lie algebra

♦ $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: Cartan decomposition

matrix of trace zero = antisymmetric matrix + symmetric matrix

♦ G adjoint group of $\mathfrak{g}$ (conjugation) BAB^{-1}

♦ $K \subseteq G$:maxl compact $Lie(K) = \mathfrak{k}$

♦ $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$: Complexification

♦ $K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$: $Lie(K_{\mathbb{C}}) = \mathfrak{k}_{\mathbb{C}}$

♦ e nilpotent in $\mathfrak{p}_{\mathbb{C}}$

$K_{\mathbb{C}}$ preserves $\mathfrak{p}_{\mathbb{C}}$.

The Kostant-Sekiguchi correspondence

There is a 1-1 correspondence between real

G-nilpotent orbits in $\mathfrak{g}$ and complex $K_{\mathbb{C}}$-

in nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$ [**J. Sekiguchi,1987**] Related orbits are diffeomorphic [**Vergne,1995**]

TECHNIQUE: Questions about real nilpotent orbits of G are studied via Complex nilpotent orbits of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$.

The $K_{\mathbb{C}}$-Prehomogeneous Spaces on $\mathfrak{p}_{\mathbb{C}}$

A triple (x, e, f) in $\mathfrak{g}_{\mathbb{C}}$ is called a *standard triple* if $[x, e] = 2e$, $[x, f] = -2f$ and $[e, f] = x$. If $x \in \mathfrak{k}_{\mathbb{C}}$ and e and $f \in \mathfrak{p}_{\mathbb{C}}$ then (x, e, f) is a *normal* triple. The action of ad_x determines a grading

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^i$$

where $\mathfrak{g}_{\mathbb{C}}^i = \{Z \in \mathfrak{g}_{\mathbb{C}} : [x, Z] = iz\}$.

$\mathrm{Lie}(G_{\mathbb{C}}^0) = \mathfrak{g}_{\mathbb{C}}^0$

Let $G_{\mathbb{C}}^0 \cap K_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}^i \cap \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}^i \cap \mathfrak{p}_{\mathbb{C}}$ by $K_{\mathbb{C}}^0$, $\mathfrak{k}_{\mathbb{C}}^i$ and $\mathfrak{p}_{\mathbb{C}}^i$ respectively. We can show that for $i \neq 0$, $(K_{\mathbb{C}}^0, \mathfrak{k}_{\mathbb{C}}^i)$ and $(K_{\mathbb{C}}^0, \mathfrak{p}_{\mathbb{C}}^i)$ are prehomogeneous spaces.

(Consequence of a Lemma of È. B. Vinberg)

Algorithm for Inner-Type Real Algebras

• Algorithm works for Matrices and exceptional algebras

• Algorithm is described in terms of *roots and weights*

(Think of functionals of the space of diagonal matrices)

Input: $\beta_i(x)$ for $(i = 1, \dots, l)$ $(K_{\mathbb{C}})$ (Djokovíc)

Step 1. Compute $\alpha_i(x)$ for $(i = 1, \dots, l)$ $(G_{\mathbb{C}})$

Step 2. Make a list $\mathcal{L}$ of all positive roots δ of $\mathfrak{g}_{\mathbb{C}}$ such that $\delta(x) = d$

Step 3. For each $\delta \in \mathcal{L}$ do

• if $\delta \in \mathfrak{p}_{\mathbb{C}}$ (respectively $\mathfrak{k}_{\mathbb{C}}$) set $Y_\delta = X_\delta$

[Now $\{Y_\delta\}_{\delta\in\mathcal{L}}$ is a basis for $\mathfrak{p}^d_{\mathbb{C}}$ (repectively $\mathfrak{k}^d_{\mathbb{C}}$)]

Step 4. For each $\delta \in \mathcal{L}$ check whether $[X_{\beta_i}, Y_\delta] = 0$ $\forall i | \beta_i(x) = 0$

if not delete δ from $\mathcal{L}$

[Now $\mathcal{L}$ is the list of $K^0_{\mathbb{C}}$ highest weights in $\mathfrak{p}^d_{\mathbb{C}}$ expressed in the $(\alpha_1 \dots \alpha_l)$ basis]

Step 5. Use the Cartan matrix of $\mathfrak{k}_{\mathbb{C}}$ to express the highest weights in the fundamental basis $(\omega_1, \dots \omega_l)$ of $\mathfrak{k}_{\mathbb{C}}$.

End.

Implementation - Correctness - Complexity

• Implementation in Computer algebra system LiE on a Macintosh iMac 1Ghz PowerPC G4 with 1GB DDR SDRAM.

• Correcteness - Step 4 - provided by the highest weight Theory

• Complexity - Very fast - LiE internal routines are very fast because LiE is not a multipurpose system. $\mathcal{O}(l * n^3)$ where n is the number of positive roots of $\mathfrak{g}_{\mathbb{C}}$.

• Output: a set of LaTeXcommands that can be pasted directly into LaTeX's *longtable* environment.

Example

Nilpotent orbits in type $G_{2(2)}$				
Orbit	$K_{\mathbb{C}}$-Label	i	$\dim \mathfrak{p}^i_{\mathbb{C}}$	Highest weights of $\mathfrak{p}^i_{\mathbb{C}}$
1	$\underset{1}{\circ}\ \underset{1}{\circ}$	1	2	$(1,1)$ $(3,-1)$
		2	1	$(3,1)$
2	$\underset{1}{\circ}\ \underset{3}{\circ}$	1	1	$(-1,1)$
		2	1	$(1,1)$
		3	1	$(3,1)$
3	$\underset{2}{\circ}\ \underset{2}{\circ}$	2	2	$(1,1)$ $(3,-1)$
		4	1	$(3,1)$
4	$\underset{0}{\circ}\ \underset{4}{\circ}$	2	4	$(3,1)$
5	$\underset{4}{\circ}\ \underset{8}{\circ}$	2	2	$(-1,1)$ $(3,-1)$
		6	1	$(1,1)$
		10	1	$(3,1)$

All prehomogeneous spaces are trivial except that for orbit 4, $\mathfrak{p}^2_{\mathbb{C}} = V^{3\omega_1} \simeq S^3(\mathbb{C}^2)$ which has dimension 4.

LiE Session

Here is the LiE code that calls the subroutine in order to generate the irreducible modules. This is the content of the file *rgmodule*.

```
g = G2;
rirrdmodule("\typeg",2, [1,1],[1,1], [[1,0],[3,2]], [[2,0],[-3,1]],2,g);
rirrdmodule("\typeg", 2,[1,3],[2,3], [[1,0],[3,2]], [[2,0],[-3,1]],2,g);
rirrdmodule("\typeg", 2,[2,2],[2,2], [[1,0],[3,2]], [[2,0],[-3,1]],2,g);
rirrdmodule("\typeg", 2, [0,4],[2,4], [[1,0],[3,2]], [[2,0],[-3,1]],2,g);
rirrdmodule("\typeg", 2, [4,8],[6,8], [[1,0],[3,2]], [[2,0],[-3,1]],2,g);
```

```
LiE version 2.2 created on Nov 22 1997 at 16:50:29
Authors: Arjeh M. Cohen, Marc van Leeuwen, Bert Lisser.
Mac port by S. Grimm
Public distribution version

type '?help' for help information
type '?' for a list of help entries.
> read rphmod
> read rgmodule
     \hline 1. & \typeg{1}{1} & 1 & 2 & \ba(1,1)
 \\ (3,-1) \\ \ea \\
     \cline{3-\tablewidth} & & 2 & 1 & \ba(3,1)
  \\ \ea \\
     \hline 2. & \typeg{1}{3} & 1 & 1 & \ba(-1,1)
  \\ \ea \\
   \cline{3-\tablewidth} & & 2 & 1 & \ba(1,1)
  \\ \ea \\
  \cline{3-\tablewidth} & & 3 & 1 & \ba(3,1) \\ \ea \\
     \hline 3. & \typeg{2}{2} & 2 & 2 & \ba(1,1) \\ (3,
   \\ \ea \\
     \cline{3-\tablewidth} & & 4 & 1 & \ba(3,1)
    \\ \ea \\
     \hline 4. & \typeg{0}{4} & 2 & 4 & \ba(3,1) \\ \ea
     \hline 5. & \typeg{4}{8} & 2 & 2 & \ba(-1,1) \\ (3
    \\ \ea \\
     \cline{3-\tablewidth} & & 6 & 1 & \ba(1,1) \\  \ea
     \cline{3-\tablewidth} & & 10 & 1 & \ba(3,1) \\
    \ea \\
    >
```