The Uncontroversial Mathematics Behind Garrett Lisi's Controversial "Theory of Everything" In Memory of My Grandmother Vilicia Auguste (1914 - 2008)

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July 26, 2008

Overview of the talk

- The Four Fundamental Forces
- The Grand Unified Theories
- The Standard Model
- A Theory of Everything (TOE)
- E_8
- Triality
- Structure Theory for E_8 and an "Embedding" of the Standard Model
- Kostant's general comment on E_8
- Some personal comments

The Four Fundamental Forces of Nature (csep10.phys.utk.edu/astr162/lect/cosmology/forces.html)

- **The strong interaction** is very strong, but very short-ranged. It acts only over ranges of order 10^{-13} centimeters and is responsible for holding the nuclei of atoms together. It is basically attractive, but can be effectively repulsive in some circumstances.
- **The electromagnetic force** causes electric and magnetic effects such as the repulsion between like electrical charges or the interaction of bar magnets. It is long-ranged, but much weaker than the strong force.
- **The weak force** is responsible for radioactive decay and neutrino interactions. It has a very short range and, as its name indicates, it is very weak.
- **The gravitational force** is weak, but very long ranged. Furthermore, it is always attractive, and acts between any two pieces of matter in the Universe since mass is its source.

Grand Unified Theories (csep10.phys.utk.edu/astr162/lect/cosmology/forces.html)

Spontaneous Symmetry Breaking.

There is a rather strong belief (although it is yet to be confirmed experimentally) that in the very early Universe when temperatures were very high compared with today, the weak, electromagnetic, and strong forces were unified into a single force. Only when the temperature dropped did these forces separate from each other, with the strong force separating first and then at a still lower temperature the electromagnetic and weak forces separating to leave us with the 4 distinct forces that we see in our present Universe.

The GrandUnifiedTheory is a vision of a physics theory that can combine three of the four fundamental forces into one single equation. (Strong, Weak, Electromagnetic)

Grand Unified Theories (csep10.phys.utk.edu/astr162/lect/cosmology/forces.html)

At the highest energies achievable with present-day particles accelerators (which correspond to temperatures of around 10^{15} Kelvin), the weak and electromagnetic forces lose their separate identities and merge into a single electro-weak force. According to what are termed Grand Unified Theories (GUTs), the strong and electro-weak forces will behave as a single unified force at particle energies and temperatures that are about a trillion times higher still (this is far beyond anything that can be attained on Earth). http://www.geocities.com/angolano/Astronomy/FundamentalForces.html

The Standard Model (http://en.wikipedia.org/wiki/Standard_model)

The Standard Model of particle physics is a theory that describes three of the four known fundamental interactions among the elementary particles that make up all matter. It groups the electroweak theory and quantum chromodynamics into a structure denoted by the gauge group $SU(3) \times SU(2) \times U(1)$. It is a relativistic quantum field theory which is consistent with both quantum mechanics and special relativity. To date, almost all experimental tests of the three forces described by the Standard Model have agreed with its predictions.

The formulation of the unification of the electromagnetic and weak interactions in the Standard Model is due to Steven Weinberg, Abdus Salam and, subsequently, Sheldon Glashow. The unification model was initially proposed by Steven Weinberg in 1967.

The Standard Model (More on the Gauge Group) http://math.ucr.edu/home/baez/week253.html

What people usually call the gauge group of the Standard Model:
$SU(3) \times SU(2) \times U(1)$
actually has a bit of flab in it: there's a normal subgroup that acts trivially on all known particles. This subgroup is isomorphic to $\mathbb{Z}/6\mathbb{Z}$. If we mod out by this, we get the "true" gauge group of the Standard Model:
$G = (SU(3) \times SU(2) \times U(1))/(\mathbb{Z}/6\mathbb{Z})$
And, this turns out to have a neat description. It's isomorphic to the subgroup of SU(5) consisting of matrices like this: $g = \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix}$
where G is a 3 by 3 block and H is a 2 by 2 block. For obvious reasons, Baez calls this group $S(U(3) \times U(2))$
If you want some intuition for this, think of the 3 by 3 block as related to the strong force, and the 2 by 2 block as related to the electroweak force.

TOE (http://en.wikipedia.org/wiki/Theory_of_everything)

In current mainstream physics, a Theory of Everything would unify all the fundamental interactions of nature, which are usually considered to be four in number: gravity, the strong nuclear force, the weak nuclear force, and the electromagnetic force. Because the weak force can transform elementary particles from one kind into another, the TOE should yield a deep understanding of the various different kinds of particles as well as the different forces.

A small number of scientists claim that Gödel's incompleteness theorem proves that any attempt to construct a TOE is bound to fail. Gödel's theorem states that any non-trivial mathematical theory that has a finite description is either inconsistent or incomplete. In his 1966 book The Relevance of Physics, Stanley Jaki pointed out that, because any "theory of everything" will certainly be a consistent non-trivial mathematical theory, it must be incomplete. He claims that this dooms searches for a deterministic theory of everything.

TOE (http://en.wikipedia.org/wiki/Theory_of_everything)

Freeman Dyson, Stephen Hawking agree.
But since most physicists would consider the statement of the underlying rules to suffice as the definition of a "theory of everything", these researchers argue that Gödel's Theorem does not mean that a TOE cannot exist.

On November 6, 2007 an unemployed physicist Dr. Anthony Garrett Lisi (born January 24, 1968) posted a paper "An Exceptionally Simple Theory of Everything" on the Archives.
http://en.wikipedia.org/wiki/Antony_Garrett_Lisi
Claims:

- A Theory of Everything
- gauge group E_8
- Triality

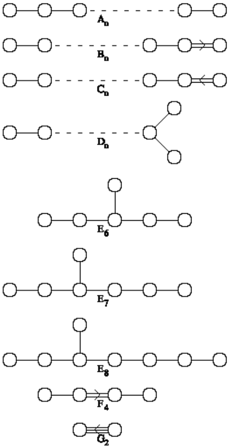
E_8 (http://en.wikipedia.org/wiki/Lie_group#Examples_of_Lie_gr

In mathematics, a Lie group (pronounced /'li:/, sounds like "Lee"), is a group which is also a differentiable manifold, with the property that the group operations are compatible with the smooth structure. They are named after the nineteenth century Norwegian mathematician Sophus Lie, who laid the foundations of the theory of continuous transformation groups.

Lie's *idée fixe* was to develop a theory of symmetries of differential equations that would accomplish for them what Évariste Galois had done for algebraic equations: namely, to classify them in terms of group theory. Additional impetus to consider continuous groups came from ideas of Bernhard Riemann, on the foundations of geometry, and their further development in the hands of Klein.

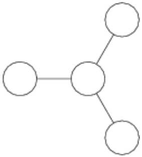
E_8 Largest simple exceptional Lie group with dimension 248

The simple complex Lie groups are the building blocks of a vast theory. They were classified by the German Wilhelm Killing circa 1880. Here is a representation through Dynkin diagrams.



Triality (http://en.wikipedia.org/wiki/Triality)

In mathematics, *triality* is a peculiar property of the group Spin(8), the double cover of 8-dimensional rotation group $D_4 = SO(8)$. Of all simple Lie groups, Spin(8) has the most symmetrical Dynkin diagram. The diagram has four nodes with one node located at the center, and the other three attached symmetrically. The symmetry group of the diagram is the symmetric group S_3 which acts by permuting the three legs. This gives rise to an S_3 group of outer automorphisms of $Spin(8)$. This automorphism group permutes the three 8-dimensional irreducible representations of $Spin(8)$; these being the vector representation and two chiral spinor representations. As such, these automorphisms do not project to automorphisms of $SO(8)$.



Structure Theory for E_8 and an "Embedding" of the Standard Model

Most of this was learned from B. Kostant either from direct conversations at MIT last year or from his talks at the MIT Lie group seminar and at his 80th birthday Conference in Vancouver at the University of British Columbia last May. There are also the video of a talk that he gave at Riverside CA and some notes that were taken by John Baez.
http://math.ucr.edu/home/baez/kostant/
Interesting discussions can be found here also:
http://golem.ph.utexas.edu/category/2008/02/kostant_on_e8.html

Structure Theory for E_8 and an "Embedding" of the Standard Model

E_8 is the largest exceptional group. Much information can be obtained from its Lie algebra $\mathfrak{e}_8$ which has the following root decomposition:

$$e_8 = X_{-\alpha_{120}} \oplus \cdots \oplus X_{-\alpha_1} \oplus \mathfrak{h} \oplus X_{\alpha_1} \oplus \cdots \oplus X_{\alpha_{120}}$$

where $\mathfrak{h}$ is maximal Abelian subalgebra of dimension 8 and each $X_{\alpha_{\pm i}}$ is a one-dimensional space contributed by the root $\alpha_{\pm i}$ a linear map from $\mathfrak{h}$ to the set of complex numbers $\mathbb{C}$. (240 roots)

Hence the dimension of E_8 is (by definition) that of $\mathfrak{e}_8$ which is 248.

In Lisi's theory each root corresponds to one elementary particle. There are 222 known elementary particles and therefore 18 would be left to discover!!

Structure Theory for E_8 and an "Embedding" of the Standard Model

E_8 has finite subgroup called the Dempwolf group (F_{Demp}). Here is another important decomposition of $\mathfrak{e}_8$:

$$\mathfrak{e}_8 = (\mathfrak{so}_8 \oplus \mathfrak{so}_8) \oplus (V_8 \otimes V_8) \oplus (S_8^+ \otimes S_8^+) \oplus (S_8^- \otimes S_8^-)$$

$$\dim E_8 = (28 + 28) + (8 \times 8) + (8 \times 8) + (8 \times 8) = 248$$

V_8, S_8^+, S_8^- are 8-dimensional *"Vector"*, *"Right-handed spinor"*, *"Left-handed spinor"* representations of $Spin(8)$ respectively. Remember these representations are related by triality.

Lisi's use of Triality

"Most GUT models require a threefold replication of the matter fields and as such, do not explain why there are three generations of fermions. Most GUT models also do not explain the little hierarchy between the fermion masses for different generations."
http://en.wikipedia.org/wiki/Grand_Unified_Theory

F_{Demp} permutes the three 64 dimensional subspaces $(V_8 \otimes V_8), (S_8^+ \otimes S_8^+)$ and $(S_8^- \otimes S_8^-)$. This is the triality result that Lisi used to construct the three generations of fermions.

This seems to be very controversial from the point of view of Physicists. See Distler's blog:
http://golem.ph.utexas.edu/˜distler/blog/archives/001505.html

An interesting comment made by Lisi on Distler's blog

"The $so(7,1) + so(8)$ acts on the $(S_8^+ \otimes S_8^+)$ as the first generation of fermions. That part works great. The structure of E_8 suggests that the second and third generations relate to the triality partners of the first, $(S_8^- \otimes S_8^-)$ and $(V_8 \otimes V_8)$, but I dont understand this relationship yet. As you know, and as I described in the paper, these second and third triality partners cannot literally be the second and third generation particles as the theory is currently constructed the relationship is merely suggestive, and I suspect something more interesting is going on. I will probably end up using a slightly different (non-triality) assignment of the fermions, and may even end up using a different group for gravity. Or I might not be able to get it to work. I've tried to be very clear, both in the paper and to the press, that this idea is still in development. Most physicists seem to understand this theory is work in progress, and treat it accordingly but thank you for spending the time to elucidate this fact so that others will understand."

Posted by: Garrett on November 23, 2007 11:08 AM

Structure Theory for E_8 and an "Embedding" of the Standard Model

There is an element a_{11} of order 11 in E_8 such that the centralizer of a_{11} in E_8 (The set of elements in E_8 such that $ga_{11}g^{-1} = a_{11}$) is:

$$S(U(3) \times U(2)) \times S(U(3) \times U(2))$$

a product of two copies of the gauge group of the Standard Model.

Remember that $S(U(3) \times U(2))$ is a subgroup of $SU(5)$. Here is a way to find two conjugate copies of $\mathfrak{su}(5)$ in $\mathfrak{e}_8$.

A product of two copies of $SU(5)$ sitting inside E_8

There is a copy of $\mathcal{G} = (\mathbb{Z}/5\mathbb{Z})^3$ in E_8. As a vector space over $(\mathbb{Z}/5\mathbb{Z})$, $\mathcal{G}$ contains exactly 31 lines. Let τ be one of such lines. Then the centralizer of τ in E_8 is

$$C = (SU(5) \times SU(5))/(\mathbb{Z}/5\mathbb{Z})$$

with dimension 48. C has a 248-dimensional representation on $\mathfrak{e}_8$. So we need to account for a 200-dimensional complement. Hence

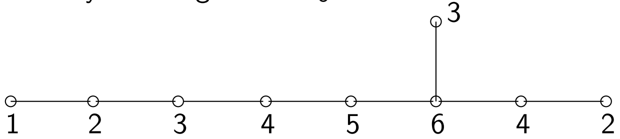
$$\mathfrak{e}_8 = (\mathfrak{su}(5) \oplus \mathfrak{su}(5)) \oplus (5 \otimes 10) \oplus (\bar{5} \otimes 10) \oplus (5 \otimes \bar{10}) \oplus (\bar{5} \otimes \bar{10})$$

where 5 is the defining representation of $SU(5)$, 10 is its second exterior power, and $\bar{5}$ and $\bar{10}$ are the duals of these.

“Is any of this useful in particle physics? In particular, what could the "doubling" mean?” (John Baez)

The two copies of $SU(5)$ are conjugate under an automorphism of E_8

Consider the Affine Dynkin diagram of E_8



Since $\mathcal{G} = SU(5) \times SU(5))/(\mathbb{Z}/5\mathbb{Z})$ is the centralizer of an element of order 5 and E_8 is adjoint, the Borel - de Siebenthal theory tells us that the conjugacy classes with semisimple centralizer are indexed by the nodes on the affine Dynkin diagram, with the centralizer given by omitting the node, and the order of the conjugacy class in the adjoint group given by the usual label on the node. So we must remove the node labelled with 5 **(Kostant's Vancouver talk May 2008)**. Hence the diagram breaks into two A_4 pieces corresponding to the two copies of $SU(5)$.

The two copies of $SU(5)$ are conjugate under an automorphism of E_8

The two copies of the gauge group of Standard Model $S(U(3) \times U(2))$ are given by the diagrams generated by the nodes labelled 1, 3, 4 in the first A_4 piece and the ones labelled (6,3,2) in the second A_4 piece.

An argument of Allen Knutson shows that the two copies of $SU(5)$ are conjugate by an element in E_8. See http://golem.ph.utexas.edu/category/2008/02/kostant_on_e8.html

Knutson's argument

"Let τ be such an element. Then τ^2 is another special element of order 5. Hence there exists a g such that $g\tau g^-1 = \tau^2$. Plainly g conjugates $C_G(\tau)$ into $C_G(\tau^2)$, which is again $C_G(\tau)$ since τ, τ^2 generate the same $\mathbb{Z}_5$.

We can distinguish the two factors of $C_G(\tau)$ by looking at the projection of τ into them; in one case we get the generator of $SU(5)$'s center (or we get its inverse, depending on identification), and in the other we get the square (or its inverse).

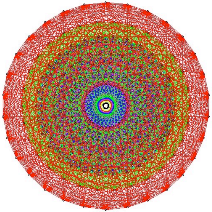
This rule reverses when we look at the projection of τ^2 instead. Summing up, conjugating by g switches the two factors of $C_G(\tau)$."

Kostant's general comment on E_8 : (Ben Wallace-Wells) THE NEW YORKER, July, 21, 2008 : http://egregium.wordpress.com/2008/07/23/bertram-kostant-on-e8/

"A word about E(8). In my opinion, and shared by others, E(8) is the most magnificent object in all of mathematics. It is like a diamond with thousands of facets. Each facet offering a different view of its unbelievable intricate internal structure. It is easy to arrive at the feeling that a final understanding of the universe must somehow involve E(8), or otherwise put, (tongue in cheek) Nature would be foolish not to utilize E(8). There was a good deal of publicity about E(8) in the last few years when a team of about 25 mathematicians, using the power of present computers and a very complicated program, succeeded in determining all of the vast number of (to use a technical term) characters associated with it. Incidentally, one of the main leaders of the team was an ex-student of mine, David Vogan. It was Vogan who told me about Lisi's paper."

Some personal comments

- Lisi's is using a real form of E_8. It is now apparent that it is the split form $E_{8(8)}$. This is precisely the one whose character table was computed by the Atlas of Lie Groups and representations in March 2007: See http://www.liegroups.org/
- Many software packages are currently being used in several large research projects involving "Classification". What will be the effect of such heavy computational tools on the methodologies of Mathematical research? (Proprietary Software and Algorithms may make Verification difficult, etc ...)
- How do we insure correctness and when is an assertion really becomes a theorem? You may be surprised to know that even some of our biggest conquests could in principle be challenged. The Four Color Problem, Classification of Simple Groups, Wiles' proof of Fermat's Theorem, Perelman's proof of Poincaré's conjecture ....
- IS MATHEMATICS BECOMING AN EXPERIMENTAL SCIENCE ?



This is a picture of the root system of type E_8, projected from 8 dimensions down to 2. Picture by John Stembridge, based on a drawing by Peter McMullen.

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