NOTICED NILPOTENT ORBITS OF REAL LIE ALGEBRAS

Alfred G. Noël
Department of Mathematics
Northeastern University
Boston, MA
anoel@lynx.neu.edu

In a recent paper **[No1]**, we gave a new classification of nilpotent orbits of simple real Lie algebras. The identification of a fundamental set of *noticed* nilpotent elements of the complex symmetric space associated with the real Lie algebra was crucial because they provided a strategy for recovering all other nilpotent elements. In this work, we describe the noticed orbits of all simple real Lie algebras. As an application of the general theory, we show how to recover the nilpotent orbits of maximal dimension (the so-called principal orbits). Such a study is not done in the other classification schemes currently available in the literature.

1
Description of Noticed Orbits in Simple Real Lie Algebras

Let $\mathfrak{g}$ be a real reductive Lie algebra with adjoint group G and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Also let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Finally, let θ be the corresponding Cartan involution of $\mathfrak{g}$ and σ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with regard to $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are obtained by complexifying $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Denote by $K_{\mathbb{C}}$ the connected subgroup of the adjoint group $G_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, with Lie algebra $\mathfrak{k}_{\mathbb{C}}$.

Definition. A nilpotent element e in $\mathfrak{p}_{\mathbb{C}}$ (or its $K_{\mathbb{C}}$-orbit) is *noticed* if the only (θ, σ)-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing e (or eqivalently meeting $K_{\mathbb{C}}.e$) is $\mathfrak{g}_{\mathbb{C}}$ itself.

A Levi subalgebra l contains e if and only if $[l, l]$ does. Thus if e is noticed in l it is actually noticed in the semi-simple subalgebra $[l, l]$ for if l is (θ, σ)-stable then so is $[l, l]$, and any nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$ is noticed in any minimal (θ, σ)-stable Levi

Typeset by $\mathcal{AMS}$-TEX

subalgebra l containing it.

We know that e is noticed if and only if the reductive subalgebra $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} = 0$, where (x, e, f) is a normal triple of $\mathfrak{g}_{\mathbb{C}}$ [**No2**]. Our goal is to give a simple procedure to identify noticed nilpotent orbits of $\mathfrak{g}$. That is the ones that correspond to noticed nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$, for the Kostant-Sekiguchi correspondence gives a bijection between the G-nilpotent orbits in $\mathfrak{g}$ and the $K_{\mathbb{C}}$-nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$ [**Se**].

1.1 Simple exceptional real Lie algebras

D. Djoković [**D1, D2**] (see also [**C-Mc**]) has computed the reductive centralizer for all real nilpotent orbits in the case where $\mathfrak{g}$ is an exceptional simple real Lie algebra. The results are given in several tables, one for each algebra. Hence the noticed orbits can be easily identified from Djoković's tables. They are the ones for which $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} = 0$. A study of the tables reveals that :

$$E_{6(2)},\, E_{6(6)},\, E_{7(7)},\, E_{8(8)},\, F_{4(4)},\, G_{2(2)}$$

are the only exceptional simple real Lie algebras to admit noticed nilpotent orbits. They are *quasi-split.*

A real form, $\mathfrak{g}$, of $\mathfrak{g}_{\mathbb{C}}$ is called *quasi-split* if there is a subalgebra, $\mathfrak{b}$, of $\mathfrak{g}$ such that $\mathfrak{b}_{\mathbb{C}} = \mathfrak{b} + i\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

The following proposition characterizes quasi-split real forms.

Proposition 1.1.0 (Rothschild). $\mathfrak{g}$ *contains a regular nilpotent iff* $\mathfrak{g}$ *is quasi-split iff* $\mathfrak{g}$ *contains a regular semisimple* H *such that* ad_H *has all real eigenvalues.*

Proof. See L. Preiss Rothschild. [**Rot**]

■

An element z of $\mathfrak{g}$ is said to be regular if $\dim G{\cdot}z \geq \dim G{\cdot}y$ for all y in g.

Hence,

Theorem 1.1.1. *An exceptional simple real Lie algebra is quasi-split if and only if it contains a noticed nilpotent element.*

■

1.2 Simple classical real Lie algebras

The following theorems were proved in [**No2**].

Theorem 1.2.1. *The non-zero noticed nilpotent orbits of* $\mathfrak{sl}_n(\mathbb{R})$ *are parametrized by partitions of* n *with distinct parts. If such a partition is even then it corresponds to two orbits labeled by* I *and* II. *The algebra* $\mathfrak{su}^*_{2n}$ *has no non-zero noticed nilpotent orbit. Also* $\mathfrak{su}(p,p+1)$ *has exactly one non-zero noticed nilpotent orbit, which is parametrized by a one-row signed Young diagram of signature* $(p,p+1)$, $\mathfrak{su}(p,p)$ *has exactly two non zero noticed nilpotent orbit, each is parametrized by a one-row signed Young diagram of signature* (p,p) . *If* $|p-q| \geq 2$ *then* $\mathfrak{su}(p,q)$ *has no non-zero noticed nilpotent orbits.*

■

Theorem 1.2.2. *The non-zero noticed nilpotent orbits of* $\mathfrak{so}(p,q)$ *are parametrized by Young diagrams of signature (p,q) such that :*

(1) all rows are odd and can be repeated at most twice,

(2) two rows of the same length must have their leftmost boxes labeled by different signs,

(3) if all the rows have an even number of boxes labeled +, or all the rows have an even number of boxes labeled −, then one numeral I or II is attached.

There are no non-zero noticed nilpotent orbits in $\mathfrak{so}^*_{2n}$.

■

Theorem 1.2.3. *The non-zero noticed nilpotent orbits of $\mathfrak{sp}_n(\mathbb{R})$ are parametrized by signed Young diagrams of size $2n$ such that:*

(1) all rows are even an can be repeated atmost twice,

(2) two rows of the same length must have their leftmost boxes labeled with different signs.

There are no non-zero noticed nilpotent orbits in $\mathfrak{sp}(p,q)$.

■

We have seen that the only exceptional simple real Lie algebras admitting nilpotent orbits are the quasi-split ones. This is not true in the case of the classical simple real Lie algebras, as the following example indicates.

Let $\mathfrak{g} = \mathfrak{so}(6,3)$. Then from theorem 1.2.2, the orbit parametrized by the signed Young diagram:

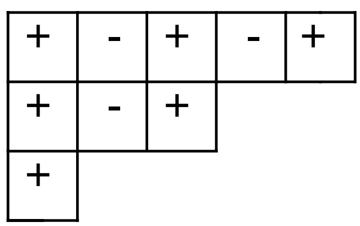


figure 1

is noticed but $\mathfrak{g}$ is not quasi-split.

2
Noticed Principal Orbits in Simple Real Lie Algebras

We shall investigate the behaviour of a principal nilpotent element of $\mathfrak{p}_{\mathbb{C}}$. Such an element and its orbit under the action of $K_{\mathbb{C}}$ have been extensively studied by

Kostant and Rallis in their seminal 1971 paper [**K-R**]. The principal orbit plays also an important role in Representation Theory. We shall prove that the semisimple part of a minimal (θ, σ)-stable Levi subalgebra containing a principal nilpotent of $\mathfrak{p}_{\mathbb{C}}$ is the complexification of a quasi-split reductive subalgebra of $\mathfrak{g}$.

Let e be in $\mathfrak{p}_{\mathbb{C}}$. Then e is principal if and only if $K_{\mathbb{C}}{\cdot}e$ is a maximal $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$ [**K-R**] that is if and only if

$$\dim\ K_{\mathbb{C}}{\cdot}e \geq \dim\ K_{\mathbb{C}}{\cdot}e' \qquad \text{for all } e' \in \mathfrak{p}_{\mathbb{C}}.$$

If the orbit $G{\cdot}\lambda_{\mathbb{R}}$ corresponds to the orbit $K_{\mathbb{C}}{\cdot}\lambda_{\mathbb{C}}$, then

$$\dim_{\mathbb{C}} K_{\mathbb{C}}{\cdot}\lambda_{\mathbb{C}} = \tfrac{1}{2}\dim_{\mathbb{R}} G{\cdot}\lambda_{\mathbb{R}} = \tfrac{1}{2}\dim_{\mathbb{C}} G_{\mathbb{C}}{\cdot}\lambda_{\mathbb{C}}$$

Kostant and Rallis give several characterizations of a principal nilpotent of $\mathfrak{p}_{\mathbb{C}}$. We will use the following criterion due to them. We will say that e is principal if and only if

$$\dim\ K_{\mathbb{C}}{\cdot}e = \dim\ \mathfrak{p}_{\mathbb{C}} \text{ - } \dim\ \mathfrak{a}_{\mathbb{C}},$$

where $\mathfrak{a}_{\mathbb{C}}$ is the complexification of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. The dimension of $\mathfrak{a}$ is called the *real rank* of $\mathfrak{g}$ (see Helgason [**He**]). The following theorem gives a characterization of the *quasi-split* simple real Lie algebras $\mathfrak{g}$ in terms of the noticed principal elements.

Theorem 2.1. *Let $\mathfrak{g}$ be a simple real Lie algebra. Then $\mathfrak{g}$ is quasi-split if and only there exists a nilpotent element $\lambda_{\mathbb{C}}$ of $\mathfrak{p}_{\mathbb{C}}$ such that $\lambda_{\mathbb{C}}$ is noticed and principal.*

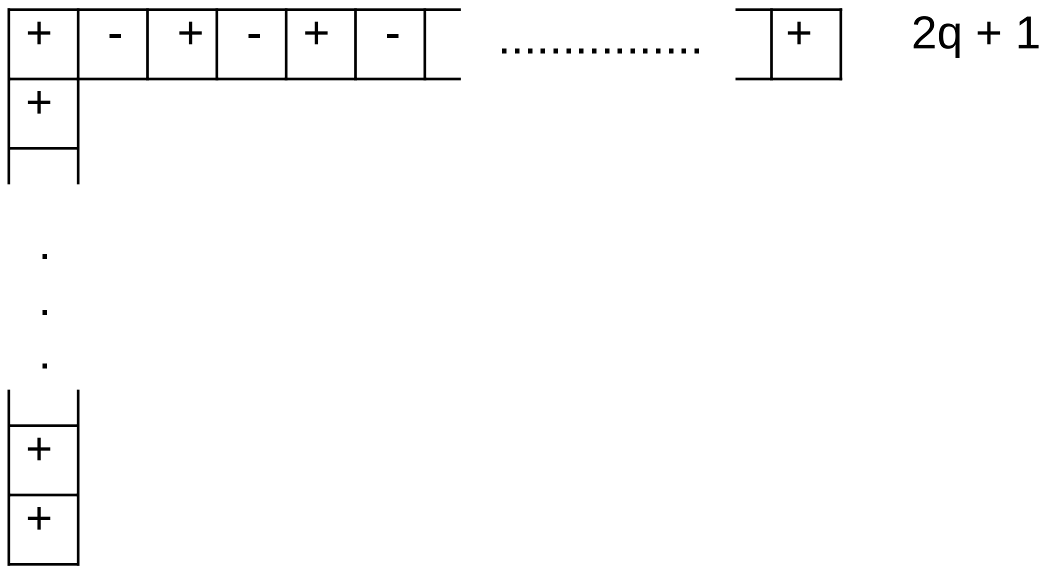
Proof. See [**No2**]

■

Next we describe the principal nilpotent orbits of the non quas-split simple real Lie algebras.

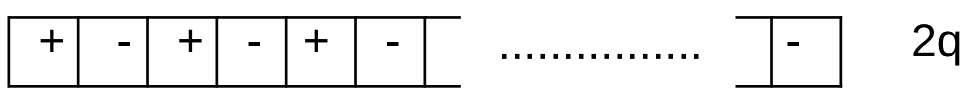
Algebras **Principlal nilpotent orbits**

$\mathfrak{su}(p,q)$ $\quad p > q$
$\mathfrak{so}(p,q)$
$\mathfrak{sp}(p,q)$

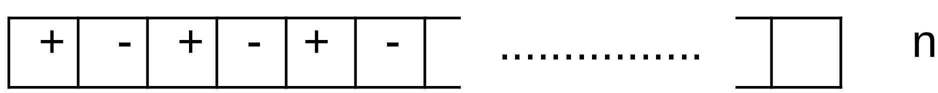


for $\mathfrak{so}(p,q)$ If q is even then there are two such orbits and the Young diagram should be labeled with Roman numerals I or II see [**C-Mc**].

$\mathfrak{sp}(q,q)$



$\mathfrak{su}^*_{2n}$
$\mathfrak{so}^*_{2n}$



for $\mathfrak{so}^*_{2n}$ If n is even then there are two such orbits [**C-Mc**].

In **[No1, No2]** we described a one-to-one correspondence between nilpotent $K_{\mathbb{C}}$-orbits on $\mathfrak{p}_{\mathbb{C}}$ and $K_{\mathbb{C}}$-conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$, where l be a (θ, σ)-stable Levi Subalgebra of $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{q}_l$ a θ-stable parabolic subalgebra of $[l, l]$ and $\mathfrak{w}_l$ a subspace of $\mathfrak{q}_l$ whose properties are described in **[No1, No2]**.

The next theorem gives some properties of the real form l_0 of the minimum (θ, σ)-stable Levi subalgebra l conaining a principal nilpotent element e in $\mathfrak{p}_{\mathbb{C}}$.

Theorem 2.2. *Let e be a nilpotent principal element of $\mathfrak{p}_{\mathbb{C}}$. Then for any normal triple (x, e, f) corresponding to a triple $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ as described in the above correspondence:*

1. *e is regular in $l \cap \mathfrak{p}_{\mathbb{C}}$*
2. *the the real form l_0 of l is quasi-split*
3. *if $\mathfrak{q}_l = m \oplus v$ is a Levi decomposition of $\mathfrak{q}_l$ then dim $m \cap \mathfrak{k}_{\mathbb{C}}$ = dim $\frac{v \cap \mathfrak{p}_{\mathbb{C}}}{[v \cap \mathfrak{k}_{\mathbb{C}}, v \cap \mathfrak{p}_{\mathbb{C}}]}$ and q_l is a Borel subalgebra of l*
4. *$\mathfrak{g}$ and l_0 have the same real rank.*

Proof. See **[No2]**

■

The previous theorem and some results of Bala and Carter **[B-C1]**, **[B-C2]** allows us to describe the type of the quasi-split real form of l. The next table gives the type of l and l_0 for the non quasi-split cases.

Table 1

Algebra	$[l,l]$	$[l_0,l_0]$	
$\mathfrak{su}(p,q)$	$\mathfrak{sl}_{2q+1}(\mathbb{C})$	$\mathfrak{su}(q+1,q)$	
$\mathfrak{su}^*_{2n}$	$\mathfrak{sl}_n(\mathbb{C})\oplus\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	
$\mathfrak{so}(p,q)$	$\mathfrak{so}_{2q+1}(\mathbb{C})$	$\mathfrak{so}(q+1,q)$	$p+q$ even
$\mathfrak{so}(p,q)$	$\mathfrak{so}_{2(q+1)}(\mathbb{C})$	$\mathfrak{so}(q+2,q)$	$p+q$ odd
$\mathfrak{so}^*_{2n}$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{su}(\frac{n}{2},\frac{n}{2}))$	n even
		$\mathfrak{su}(\frac{n+1}{2},\frac{n-1}{2})$	n odd
$\mathfrak{sp}(p,q)$	$\mathfrak{sl}_{2q+1}(\mathbb{C})$	$\mathfrak{su}(q+1,q)$	$p>q$
$\mathfrak{sp}(q,q)$	$\mathfrak{sl}_{2q}(\mathbb{C})$	$\mathfrak{su}(q,q)$	
$E_{6(-14)}$	$\mathfrak{sl}_5(\mathbb{C})$	$\mathfrak{su}(3,2)$	
$E_{6(-26)}$	$\mathfrak{sl}_3(\mathbb{C})\oplus\mathfrak{sl}_3(\mathbb{C})$	$\mathfrak{sl}_3(\mathbb{C})$	
$E_{7(-5)}$	E_6	$E_{6(2)}$	
$E_{7(-25)}$	$\mathfrak{sl}_6(\mathbb{C})$	$\mathfrak{su}(3,3)$	
$E_{8(-24)}$	E_6	$E_{6(2)}$	
$F_{4(-20)}$	$\mathfrak{sl}_3(\mathbb{C})$	$\mathfrak{su}(2,1)$	

References

[B-C1] P. Bala and R. Carter, *Classes of unipotent elements in simple algebraic groups I*, Math. Proc. Camb. Phil. Soc. **79** (1976), 401-425.

[B-C2] P. Bala and R. Carter, *Classes of unipotent elements in simple algebraic groups II*, Math. Proc. Camb. Phil. Soc. **80** (1976), 1-18.

[C-Mc] D. H. Collingwood, W. M. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reihnhold Mathematics Series, New York, 1993.

[D1] D. Djoković, *Classification of nilpotent elements in simple exceptional real Lie algebras of inner type and description of their centralizers*, J. Alg. **112** (1988), 503-524.

[D2] D. Djoković, *Classification of nilpotent elements in the simple real Lie algebras $E_{6(6)}$ and $E_{6(-26)}$ and description of their centralizers*, J. Alg. **116** (1988), 196-207.

[Dy] E. Dynkin, *Semisimple subalgebras of simple Lie algebras*, Amer. Soc. Transl. Ser. 2 **6,** (1957), 111-245.

[He] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[Ki] D. R. King, *The Component Groups of Nilpotents in Exceptional Simple Real Lie Algebras* , Communications In Algebra **20 (1)** (1992), 219-284.

[K-R] B. Kostant, S. Rallis, *Orbits and Representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753-809.

[No1] A. G. Noël, *Classification of Nilpotent Orbits in Symmetric Spaces, Proceedings of the DIMACS workshop for African-American Researchers in the Mathematical Sciences*, Amer. Math. Soc. (To appear).

[No2] A. G. Noël, *Nilpotent Orbits and θ-stable parabolic subalgebras , Ph. D. Thesis*, Northeastern University, Boston (March 1997).

[Rot] L. P. Rothschild, *Orbits in a real reductive Lie algebra*, Trans. Amer. Math. Soc. **168** (June 1972), 403-421.

[Se] J. Sekiguchi, *Remarks on a real nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **Vol. 39, No. 1** (1987), 127-138.

[Vo] D. A. Vogan, *Representation of Real Reductive Lie Groups*, vol. **15**, Birkhäuser, Progress in Mathematics Boston, 1981.