Centralizers of Nilpotents and The Bala-Carter Classification

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Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra with adjoint group $G_{\mathbb{C}}$ and e a nilpotent element in $\mathfrak{g}_{\mathbb{C}}$.

Let $G_{\mathbb{C}}^e$ centralizer of e in $G_{\mathbb{C}}$. $G_{\mathbb{C}}^e$ is not connected in general. Often it is required to compute the finite group

$$A(e) = \frac{G_{\mathbb{C}}^e}{(G_{\mathbb{C}}^e)_\circ}$$

where $(G_{\mathbb{C}}^e)_\circ$ is the identity component.

E. Sommers (1997) gave a unified description of the conjugacy classes of $A(e)$.

A pseudo-Levi subalgebra l of $\mathfrak{g}_{\mathbb{C}}$ is defined to be the centralizer in $\mathfrak{g}_{\mathbb{C}}$ of a semisimple element z of $G_{\mathbb{C}}$. (Notation $l = \mathfrak{g}_{\mathbb{C}}^{z}$)

Pseudo-Levi subgroups L of $G_{\mathbb{C}}$ are of the form $(G_{\mathbb{C}}^{z})_{\circ}$ the identity component of the centralizer of z in $G_{\mathbb{C}}$.

Let Z be the center of L in $G_{\mathbb{C}}$ then the group $\frac{Z}{Z_{\circ}}$ is cyclic. (Sommers)

A nilpotent element e of $\mathfrak{g}_{\mathbb{C}}$ is called **distinguished** if the conditions $x \in \mathfrak{g}_{\mathbb{C}}$ semisimple and $[x, e] = 0$ imply that x is in the center of $\mathfrak{g}_{\mathbb{C}}$.

Let $L \subset G_{\mathbb{C}}$ be a pseudo-Levi subgroup with center Z and Lie algebra l a pseudo-Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Given a conjugacy class $\bar{c}$ of $A(e)$, l has the **key property** for $(e, \bar{c})$ if $e \in l$ and there exits $z \in Z$ such that

1. $zZ_\circ$ generates the cyclic group $\frac{Z}{Z_\circ}$

2. $z(G_{\mathbb{C}}^e)_\circ = \bar{c}$

Moreover if l is a minimal pseudo-Levi subalgebra with the key property for $(e, \bar{c})$ then e is distinguished in l.

Sommers shows how to build minimal pseudo-Levi subalgebras with the key property for a given pair $(e, \bar{c})$.

Theorem (Sommers). There is a bijection Φ between $G_{\mathbb{C}}$ conjugacy classes of pairs (l, e), where l is a pseudo-Levi subalgebra and e is a distinguished nilpotent in l, and $G_{\mathbb{C}}$ conjugacy classes of pair $(e, \bar{c})$, where e is a nilpotent element in $\mathfrak{g}_{\mathbb{C}}$ and $\bar{c}$ is a conjugacy class of $A(e)$

Example. Let $\mathfrak{g}_{\mathbb{C}}$ be $\mathfrak{so}_5(\mathbb{C})$, the complex Lie algebra of 5×5 orthogonal matrices. It is known that its nilpotent classes are parametrized by certain partitions of 5. Consider the class [3,1,1]. A computation shows that the only two pseudo-Levi subalgebras (up to conjugacy) containing a nilpotent e in that class are of the form $\mathfrak{sl}_2(\mathbb{C})$ and $2\mathfrak{sl}_2(\mathbb{C})$.

Sommer's Correspondence

$(e, 1) \rightleftarrows (\mathfrak{sl}_2(\mathbb{C}), e)$

$(e, -1) \rightleftarrows (2\mathfrak{sl}_2(\mathbb{C}), e)$

In this case $A(e) = \mathbb{Z}_2$.

Sommer's theorem is an extension of the **Bala-Carter** classification which states:

The nilpotent orbits $G_{\mathbb{C}}.e$ of $\mathfrak{g}_{\mathbb{C}}$ are in bijection with $G_{\mathbb{C}}$ conjugacy classes of pairs (l, p_l) where l is a Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ in which e is distinguished and p_l a distinguished parabolic subalgebra of $[l, l]$.

In fact $(e, 1)$ always goes to the B-C Levi l.

Our goal is to extend Sommer's result to real reductive Lie groups.

Let $\mathfrak{g}$ be a real reductive Lie algebra with adjoint group G and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ relative to a Cartan involution Θ.

We complexify $\mathfrak{g}$ to obtain $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$.

Denote by σ the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with regard to $\mathfrak{g}$.

Let $K_{\mathbb{C}}$ be the complexification of K the connected subgroup of G with Lie algebra $\mathfrak{k}$. $K_{\mathbb{C}}$ preserves $\mathfrak{p}_{\mathbb{C}}$ under the adjoint action.

Sekiguchi proves that there is a one to one correspondence between nilpotent G-orbits in $\mathfrak{g}$ and nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$.

Therefore it is reasonable to solve the following problem.

Let e be a nilpotent element in $\mathfrak{p}_{\mathbb{C}}$ and $A_k(e) = \frac{K_{\mathbb{C}}^e}{(K_{\mathbb{C}}^e)_\circ}$. Give a unified description of the elements of $A_k(e)$.

A pseudo-Levi subalgebra l of $\mathfrak{g}_{\mathbb{C}}$ is said to be **elliptic** if it is the centralizer in $\mathfrak{g}_{\mathbb{C}}$ of an elliptic element of $K_{\mathbb{C}}$.

The connected subgroup L of $G_{\mathbb{C}}$ with Lie algebra l is an elliptic pseudo Levi subgroup of $G_{\mathbb{C}}$. Let Z be the center of L then:

Non trivial Fact:

$$\frac{Z \cap K_{\mathbb{C}}}{(Z \cap K_{\mathbb{C}})_{\circ}} \quad \textbf{is cyclic}$$

A nilpotent element e of $l \cap \mathfrak{p}_{\mathbb{C}}$ is called **noticed** if the conditions $z \in l \cap \mathfrak{k}_{\mathbb{C}}$ semisimple and $[z, e] = 0$ imply that z is in the center of l.

We can choose e such that it lies in a Kostant-Sekiguchi $\mathfrak{sl}_2-$triple $\{x, e, f\}$ that is $x \in \mathfrak{k}_{\mathbb{C}}$, e and f in $\mathfrak{p}_{\mathbb{C}}$, $\sigma(e) = f$.

A θ-stable parabolic subalgebra $\mathfrak{q}$ of l is said to be noticed for e if there is a K-S triple in l for which $\mathfrak{q}$ is the sum of the non-negative eigenspaces of $ad(x)$ acting on l.

Given a conjugacy class $\bar{c}$ of $A_k(e)$, l has the **key property** for $(e, \bar{c})$ if $e \in l$ and there exits $z \in Z \cap K_{\mathbb{C}}$ such that

1. $z(Z \cap K_{\mathbb{C}})_\circ$ generates $\frac{Z \cap K_{\mathbb{C}}}{(Z \cap K_{\mathbb{C}})_\circ}$

2. $z(K^e_{\mathbb{C}})_\circ = \bar{c}$

Moreover if l is a minimal elliptic pseudo-Levi subalgebra with the key property for $(e, \bar{c})$ then e is noticed in l.

We know how to build minimal eliptic pseudo-Levi subalgebras with the key property for a given pair $(e, \bar{c})$.

Theorem (1999). There is a one to one correspondence between $K_{\mathbb{C}}$-conjugacy classes of pairs $(e, \bar{z})$, where e is a nilpotent in $\mathfrak{p}_{\mathbb{C}}$, $\bar{z} \in A_k(e)$ and $K_{\mathbb{C}}$-conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w})$ where l is an elliptic pseudo-Levi subalgebra in which e is noticed, $\mathfrak{q}_l$ is a noticed parabolic of l for e, and $\mathfrak{w}$ is a certain prehomogeneous space.

Example 1. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$.
The two non-zero nilpotent orbits of $\mathfrak{g}$ are parametrized by the matrices

$$e_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix},\ e_2 = \frac{1}{2}\begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A computation gives:

$A_k(e_1) = id$ and $A_k(e_2) = \mathbb{Z}_2$

We obtain the following correspondence:

$$(e_1, id) \rightleftarrows (\mathfrak{sl}_3(\mathbb{C}), b_1, \mathfrak{w}_1)$$

$$(e_2, id) \rightleftarrows (\mathfrak{sl}_3(\mathbb{C}), b_2, \mathfrak{w}_2)$$

$$(e_2, -id) \rightleftarrows (\mathfrak{sl}_2(\mathbb{C}) \oplus V, b_3, \mathfrak{w}_3)$$

Example 2. Let $\mathfrak{g} = \mathfrak{so}(3,2)$.
The complex nilpotent orbit $[3,1,1]$ splits into three classes e_1,e_2, e_3 in $\mathfrak{g}$.

- + -	+ - +
+	+
+	-
e_1, e_2	e_3

$$(e_1, id) \rightleftarrows (\mathfrak{sl}_2(\mathbb{C}), b_1, \mathfrak{w}_1)$$

$$(e_1, -id) \rightleftarrows (2sl_2(\mathbb{C}), b_2, \mathfrak{w}_2)$$

$$(e_2, id) \rightleftarrows (\mathfrak{sl}_2(\mathbb{C}), b_3, \mathfrak{w}_3)$$

$$(e_2, -id) \rightleftarrows (2sl_2(\mathbb{C}), b_4, \mathfrak{w}_4)$$

$$(e_3, id) \rightleftarrows (\mathfrak{so}_5(\mathbb{C}), \mathfrak{q}, \mathfrak{w}_5)$$

$$(e_3, -id) \rightleftarrows (2sl_2(\mathbb{C}), b_5, \mathfrak{w}_6)$$

This theorem is an extension of the our classification which states:

The nilpotent orbits $K_{\mathbb{C}}.e$ of $\mathfrak{g}_{\mathbb{C}}$ are in bijection with $K_{\mathbb{C}}$-conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ where l is a (Θ, σ) Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ in which e is noticed, $\mathfrak{q}_l$ a noticed parabolic subalgebra of $[l, l]$ and $\mathfrak{w}_l$ is an $L \cap K_{\mathbb{C}}$-module in l.

In fact $(e, 1)$ always goes to this (Θ, σ)-stable Levi l .

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