Component Groups of Centralizers of Nilpotents in Complex Symmetric Spaces

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Given an $n \times n$ real matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

The trace of A is $tr(A) = \sum_{i=1}^{n} a_{ii}$.

Define $\mathfrak{sl}_n(\mathbb{R})$ the set of real matrices of trace zero. Similary $\mathfrak{sl}_n(\mathbb{C})$ is the set of complex matrices of trace zero.

$\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{sl}_n(\mathbb{C})$ are Lie algebras with Bracket operation : $[A, B] = AB - BA$.

A $n \times n$ matrix E is nilpotent if there exists a positive integer k such that:

$$E^k = 0$$

Let $SL_n(\mathbb{R})$ be the set of real invertible matrices with determinant 1. (n odd)

Given a nilpotent matrix E in $\mathfrak{sl}_n(\mathbb{R})$, Denote $SL_n(\mathbb{R})^E$ the set of invertible matrices that commute with E under matrix multiplication and call it the **centralizer** of E in $SL_n(\mathbb{R})$.

$SL_n(\mathbb{R})^E$ is a Lie group which may not be topologically connected . Instead it may consist of a finite number of components.

$SL_n(\mathbb{R})^E_\circ$ is the component containing the identity. Consider the quotient:

$$A(E) = \frac{SL_n(\mathbb{R})^E}{SL_n(\mathbb{R})^E_\circ}$$

Problem:
Give a unified description of the conjugacy classes of $A(E)$ for all real reductive Lie groups including the exceptional ones.

Eric Sommers (1997) gave such a description for complex Lie groups.

Application: Unitary Representation of Real Lie Groups.

Example 1.

Let $E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ be a nilpotent

matrix in $\mathfrak{sl}_3(\mathbb{R})$. Compute $SL_3(\mathbb{R})^E$.

A simple computation gives:

$$SL_3(\mathbb{R})^E = \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & \frac{1}{a^2} \end{pmatrix} \; a \neq 0, b, c, d \in \mathbb{R}$$

$$SL_3(\mathbb{R})^E = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} \\ 0 & 1 & 0 \\ 0 & da^2 & 1 \end{pmatrix}$$

Let $F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = H$$

(H, E, F) is an $\mathfrak{sl}_2$-triple.

$$SL_3(\mathbb{R})^{(H,E,F)} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix} \qquad a \neq 0.$$

$$SL_3(\mathbb{R})^E = SL_3(\mathbb{R})^{(H,E,F)} \times \text{ Unipotent}$$

$$A(E) = \frac{SL_3(\mathbb{R})^E}{SL_3(\mathbb{R})^E_\circ} = \frac{SL_3(\mathbb{R})^{(H,E,F)}}{SL_3(\mathbb{R})^{(H,E,F)}_\circ}$$

$$SL_3(\mathbb{R})^{(H,E,F)} \backsimeq \mathbb{R}^*$$

$$A(E) = \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \} = \mathbb{Z}_2$$

The above method will not work well for arbitrarily large matrices. Also it will not be appropriate for the exceptional groups.

General Strategy:

$(E, \bar{C}) \iff (l, \mathfrak{q}_l, w_l)$

E is a nilpotent in $\mathfrak{sl}_n(\mathbb{R})$

$\bar{C}$ a conjugacy class of $A(E)$

l: elliptic pseudo-Levi subalgebra of $\mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{q}_l$: a Θ-stable parabolic subalgebra of l

$\mathfrak{w}_l$: a certain subspace of $\mathfrak{q}_l$. E sits nicely in it.

The set of complex matrices of trace zero which commute with a given orthogonal A is an **elliptic pseudo-Levi** subalgebra.

Example

Let $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\mathfrak{sl}_3(\mathbb{C})^A = \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & -a-e \end{pmatrix} \; a, b, d, e \in \mathbb{C}$$

$$\mathfrak{sl}_3(\mathbb{C})^A \simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}.$$

Parabolic Subalgebras

$\mathbf{d} = [d_1, \ldots d_k]$ be a partition of n

$$E = \begin{pmatrix} N_1 & 0 & \ldots & 0 \\ 0 & N_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & N_k \end{pmatrix} \iff \mathbf{d}$$

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}_{d_i \times d_i}$$

Jordan Decomposition

Example

$\mathbf{d}$=$[4,2,1]$ a partition of 7

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$l \cong \begin{pmatrix} * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

$$\mathfrak{b} \cong \begin{pmatrix} * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathfrak{l}$: Levi Subalgebra

$\mathfrak{b}$: Parabolic Subalgebra

Levi subalgebra : abstraction of diagonal blocks

Parabolic Subalgebra : abstraction of upper triangular matrices

$\mathfrak{w}$ **tells us the place of** E **in the parabolic**

Description of additional data

$\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}_n(\mathbb{R}) \oplus$ sym real matrices

$\mathfrak{so}_n(\mathbb{R})$: $n \times n$ skew-sym real matrices

$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{so}_n(\mathbb{C}) \oplus$ sym cplx matrices

$\Theta(X) = -X^T$: Cartan involution

$\mathfrak{p}_{\mathbb{C}}$: Set of complex symmetric matrices

$SO_n(\mathbb{C})$: group of complex orthogonal matrices

$SO_n(\mathbb{C})$ preserves $\mathfrak{p}_{\mathbb{C}}$ under conjugation

Kostant-Sekiguchi Correspondence

There is a one to one correspondence between nilpotent $SL_n(\mathbb{R})$-orbits in $\mathfrak{sl}_n(\mathbb{R})$ and nilpotent $SO_n(\mathbb{C})$-orbits in $\mathfrak{p}_{\mathbb{C}}$.

$$SL_n(\mathbb{R}).E \leftrightarrows SO_n(C).e \qquad e \in \mathfrak{p}_{\mathbb{C}}$$

$$A(E) = \frac{SL_n(\mathbb{R})^E}{SL_n(\mathbb{R})^E_\circ} = \frac{SO_n(\mathbb{C})^e}{SO_n(\mathbb{C})^e_\circ} = A_k(e)$$

Equivalent Problem:

Give a unified description of $A_k(e)$ for all complex symmetric spaces.

Any nilpotent E of $\mathfrak{sl}_n(\mathbb{R})$ is $SL_n(\mathbb{R})$ conjugate to a nilpotent N which lies in a $\mathfrak{sl}_2$-triple:

(H, N, F) with $H = H^T$ and $N^T = F$

$x = i(N - F)$: skew-symmetric

$e = \frac{1}{2}(N + F + iH)$: symmetric

$f = \frac{1}{2}(N + F - iH)$: symmetric : $e = \bar{f}$

$(H, N, F) \iff (x, e, f)$

$$A(E) = A(N) = \frac{SL_n(\mathbb{R})^{(H,N,F)}}{SL_n(\mathbb{R})^{(H,N,F)}_{\circ}}$$

$$= \frac{SO_n(\mathbb{C})^{(x,e,f)}}{SO_n(\mathbb{C})^{(}(x,e,f)_{\circ}} = A_k(e)$$

Example

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$x = \sqrt{2} \begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$e = \tfrac{1}{\sqrt{2}} \begin{pmatrix} i\sqrt{2} & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -i\sqrt{2} \end{pmatrix}$$

Θ-Stable Parabolic Subalgebras

$$X \in \mathfrak{q} \iff -X^T \in \mathfrak{q}$$

$$H_1 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$E_1 = \tfrac{1}{2} \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & -1 \\ i & 1 & 0 \end{pmatrix}$$

$\mathfrak{b} = \mathbb{C}H_1 \oplus \mathbb{C}E_3 \oplus \mathbb{C}H_2 \oplus \mathbb{C}E_1 \oplus \mathbb{C}E_2$

conjugate to upper triangular matrices

Noticed nilpotent elements

Given an elliptic pseudo-Levi $l \subseteq \mathfrak{sl}_n(\mathbb{C})$ and a nilpotent e of $l \cap \mathfrak{p}_{\mathbb{C}}$ is called **noticed** if the partition associated with e is made of different parts **[Noël]**

Equivalently

A nilpotent $e \in l \cap \mathfrak{p}_{\mathbb{C}}$ is said to be noticed in l if the conditions

1. $z \in l \cap \mathfrak{so}_n(\mathbb{C})$
2. z similar to a diagonal matrix
3. z commute with e

imply that z commute with l.

Noticed nilpotents classify all nilpotents [Noël]

Theorem (1999). There is a one to one correspondence between $SO_n(\mathbb{C})$-conjugacy classes of pairs $(e, \bar{z})$, where e is a nilpotent in $\mathfrak{p}_{_\mathbb{C}}$, $\bar{z}$ a conjugacy class in $A_k(e)$ and $SO_n(\mathbb{C})$-conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ where l is an elliptic pseudo-Levi subalgebra in which e is noticed, $\mathfrak{q}_l$ is a noticed Θ-stable parabolic of l for e, and $\mathfrak{w}$ is a is prehomogeneous subspace of $\mathfrak{q}_l$.

Example 1. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$.
The two non-zero nilpotent orbits of $\mathfrak{g}$ are parametrized by the matrices

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix} \in \mathfrak{p}_{\mathbb{C}}$$

A computation gives:

$A_k(E_1) = \mathbb{Z}_2$ and $A_k(E_2) = id$

We obtain the following correspondence:

$(E_1, id) \rightleftarrows (\mathfrak{sl}_3(\mathbb{C}), \mathfrak{b}, \mathbb{C}E_1)$

$(E_1, -id) \rightleftarrows (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}, \mathfrak{b}_1, \mathfrak{w})$

$(E_2, id) \rightleftarrows (\mathfrak{sl}_3(\mathbb{C}), \mathfrak{b}, \mathbb{C}E_2)$

Example 2. Let $\mathfrak{g} = \mathfrak{sl}_5(\mathbb{R})$.

Nilpotent: $E \iff [2,2,1]$ (Not noticed)

$(E, id) \rightleftarrows (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}, \mathfrak{b}', \mathfrak{w}')$

$(E, -id) \rightleftarrows (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}), \mathfrak{b}'', \mathfrak{w}'')$

Fact

$(E, id) \rightleftarrows (l, \mathfrak{q}_l, \mathfrak{w}_l)$

l is a θ-stable Levi subalgebra. [**Noël**]

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