COMPONENT GROUPS OF CENTRALIZERS OF NILPOTENTS IN COMPLEX SYMMETRIC SPACE

DONALD R. KING
Department of Mathematics, Northeastern University, Boston, Massachusetts 02115
E-mail: donking@neu.edu

ALFRED G. NOEL
Dept of Math and Computer Science, University of Massachusetts, Boston, MA 02125
E-mail: alfred.noel@umb.edu

ABSTRACT. Let G be the adjoint group of a simple Lie algebra $\mathfrak{g}$, and let $K_{\mathbf{C}} \to Aut(\mathfrak{p}_{\mathbf{C}})$ be the complexified isotropy representation at the identity coset of the corresponding symmetric space. If $e \in \mathfrak{p}_{\mathbf{C}}$ is nilpotent, we consider the centralizer of e in $K_{\mathbf{C}}$. We show that the conjugacy classes of the component group of this centralizer can be described in terms generalizing the Bala-Carter classification of nilpotent orbits in the complexification of $\mathfrak{g}$.

1. Introduction

In [10], the second author presented a classification of nilpotent elements in a complex symmetric space which generalizes the Bala-Carter classification of nilpotent elements in a complex semi-simple Lie algebra. (By complex symmetric space we mean the complexified tangent space at the identity coset of a non compact symmetric space G/K.) Inspired by the main results of [14], we extend this classification to provide additional information about the centralizers of nilpotents in a complex symmetric space. Precise knowledge of the centralizer of such a nilpotent is needed to "quantize" its orbit, i.e., to construct a corresponding irreducible unitary representation of G. (See Theorem 8.12 of [4].)

In order to state our main results (Theorems 1 and 2 below) we require some notation and definitions. We assume that $\mathfrak{g}$ is a real simple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. θ is the associated Cartan involution. Let $\mathfrak{g}_{\mathbf{C}}$, $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{p}_{\mathbf{C}}$ denote the complexifications of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ respectively. θ extends to a complex linear involution on $\mathfrak{g}_{\mathbf{C}}$. Let σ denote conjugation relative to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbf{C}}$. The algebra $\widetilde{\mathfrak{u}} = \mathfrak{k} + i\mathfrak{p}$ is a "compact" real form of $\mathfrak{g}_{\mathbf{C}}$. τ denotes conjugation of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\widetilde{\mathfrak{u}}$. $G_{\mathbf{C}}$ is the adjoint group of $\mathfrak{g}_{\mathbf{C}}$. G, K, and $K_{\mathbf{C}}$ are the connected subgroups of $G_{\mathbf{C}}$ corresponding to the Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{k}_{\mathbf{C}}$.

Definition 1. *If e is a nilpotent element of $\mathfrak{p}_{\mathbf{C}}$, let $A(e)$ and $A_k(e)$ denote the component groups $G^e_{\mathbf{C}}/(G^e_{\mathbf{C}})_0$ and $K^e_{\mathbf{C}}/(K^e_{\mathbf{C}})_0$ respectively. $G^e_{\mathbf{C}}$ is the centralizer of e in $G_{\mathbf{C}}$. $(G^e_{\mathbf{C}})_0$ is the identity component of $G^e_{\mathbf{C}}$. $K^e_{\mathbf{C}}$ and $(K^e_{\mathbf{C}})_0$ are defined analogously.*

Proofs should be sent to Donald R. King, Department of Mathematics, Northeastern University, Boston, MA 02115.

Definition 2. *An ordered triple* $\{Z_1, Z_2, Z_3\}$ *of elements in* $\mathfrak{g}_{\mathbf{C}}$ *(or* $\mathfrak{g}$*) is said to be an* $sl(2)$*-triple if the following commutation relations are satisfied:* $[Z_1, Z_2] = 2Z_2, [Z_1, Z_3] = -2Z_3$*, and* $[Z_2, Z_3] = Z_1$*. The* $sl(2)$*-triple* $\{Z_1, Z_2, Z_3\}$ *is said to be normal if* $Z_1 \in \mathfrak{k}_{\mathbf{C}}$*, and* $Z_2,\ Z_3 \in \mathfrak{p}_{\mathbf{C}}$*.*

Let $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]$ denote the set of nilpotent elements in $\mathfrak{p}_{\mathbf{C}}$ and $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]/K_{\mathbf{C}}$ denote the set of $K_{\mathbf{C}}$ orbits in $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]$. Kostant and Rallis [9] have shown that there is a bijection between $K_{\mathbf{C}}$ conjugacy classes of normal $sl(2)$-triples of $\mathfrak{g}_{\mathbf{C}}$ and $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]/K_{\mathbf{C}}$.

Definition 3. [14] $\mathfrak{l}$ *is said to be a* **pseudo Levi subalgebra** *of* $\mathfrak{g}_{\mathbf{C}}$ *if it is the centralizer in* $\mathfrak{g}_{\mathbf{C}}$ *of a semisimple element of* $G_{\mathbf{C}}$*. A pseudo Levi subalgebra* $\mathfrak{l}$ *is said to be elliptic if it is the centralizer of an elliptic element of* $K_{\mathbf{C}}$*. A Levi subalgebra is said to be elliptic if it is the centralizer of a torus in* $\mathfrak{k}$*.*

Note that an elliptic pseudo Levi subalgebra may also be a non elliptic Levi subalgebra. For example, if $\mathfrak{g} = sl(3,\ \mathbf{R})$, then the centralizer in $sl(3,\ \mathbf{C})$ of the vector part of the fundamental Cartan subalgebra is also an elliptic pseudo Levi subalgebra.

Definition 4. *Let* $\mathfrak{l}$ *be a reductive subalgebra (resp.* θ*-stable reductive subalgebra) of* $\mathfrak{g}_{\mathbf{C}}$*. A nilpotent element* e *in* $\mathfrak{l}$ *(resp.* $\mathfrak{l} \cap \mathfrak{p}_{\mathbf{C}}$*) is said to be* **distinguished** *(resp.* **noticed***) in* $\mathfrak{l}$ *if the conditions* $z \in \mathfrak{l}$ *(resp.* $z \in \mathfrak{l} \cap \mathfrak{k}_{\mathbf{C}}$*),* z *is semisimple and* $[z,\ e] = 0$ *imply that* z *is in the center of* $\mathfrak{l}$*.*

If e *is distinguished (resp. noticed) in* $\mathfrak{l}$*, a parabolic subalgebra (resp.* θ*-stable parabolic subalgebra)* $\mathfrak{q}$ *of* $\mathfrak{l}$ *is said to be noticed for* e *if there is an* $sl(2)$*-triple (resp. normal* $sl(2)$*-triple)* $\{x,\ e,\ f\}$ *in* $\mathfrak{l}$ *for which* $\mathfrak{q}$ *is the sum of the non-negative eigenspaces of* $ad(x)$ *acting on* $\mathfrak{l}$*.* $\mathfrak{q}$ *has Levi decomposition* $\mathfrak{m} \oplus \mathfrak{u}$*, where* $\mathfrak{u}$ *(resp.,* $\mathfrak{m}$*) is the sum of the of the positive (resp., zero) eigenspaces of* $ad(x)$*.*

Remark 1. *If* $\mathfrak{l}$ *in Definition 4 is semisimple, then we have the following characterizations of distinguished and noticed nilpotents.* e *is distinguished (resp., noticed) in* $\mathfrak{l}$ *if and only if there is an* $sl(2)$*-triple (resp., normal* $sl(2)$*-triple)* $\{x,\ e,\ f\}$ *such that* $\mathfrak{l}^{\{x,\ e,\ f\}}$ *(resp.,* $(\mathfrak{l} \cap \mathfrak{k}_{\mathbf{C}})^{\{x,\ e,\ f\}}$*) equals zero.*

Our first result generalizes Theorem 14 of [14]:

Theorem 1. *There is a bijection between (1)* $K_{\mathbf{C}}$ *conjugacy classes of pairs* $(e,\ C)$*, where* e *is a nilpotent in* $\mathfrak{p}_{\mathbf{C}}$ *and* C *is a conjugacy class in* $A_k(e)$*; and (2)* $K_{\mathbf{C}}$ *conjugacy classes of triples* $(\mathfrak{l},\ \mathfrak{q},\ \overline{M_{K_{\mathbf{C}}} \cdot e})$*, where* $\mathfrak{l}$ *is an elliptic pseudo Levi subalgebra in which* e *is noticed and* $\mathfrak{q} \subset \mathfrak{l}$ *is a* θ*-stable noticed parabolic for* e*. Here* $\mathfrak{q}$ *has Levi decomposition* $\mathfrak{m} \oplus \mathfrak{u}$*;* $M_{K_{\mathbf{C}}}$ *is the connected subgroup of* L *with Lie algebra* $\mathfrak{m} \cap \mathfrak{k}_{\mathbf{C}}$*; and "* $\overline{\quad}$ *" denotes closure.*

Remark 2. *It is known from* [10] *that the set* $\overline{M_{K_{\mathbf{C}}} \cdot e}$ *in Theorem 1 is the 2-eigenspace of* x *in* $\mathfrak{l} \cap \mathfrak{p}_{\mathbf{C}}$*.*

We first give some examples of the bijection asserted in Theorem 1. In these examples, when describing the triples $(\mathfrak{l},\ \mathfrak{q},\ \overline{M_{K_{\mathbf{C}}} \cdot e})$, we will specify $\mathfrak{l}$ by giving its real form $\mathfrak{l}_{\mathbf{R}} = \mathfrak{l} \cap \mathfrak{g}$ inside $\mathfrak{g}$. We also often omit a precise description of $\mathfrak{q}$ and $\overline{M_{K_{\mathbf{C}}} \cdot e}$. For $\mathfrak{g}$ exceptional, the conjugacy classes in $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]/K_{\mathbf{C}}$ are given in [1] and [2]. We will often refer to these references collectively as Djokovic's tables.

Example 1. *Let* $\mathfrak{g} = sl(3,\ \mathbf{R})$. *There are two non-zero* K_{C} *conjugacy classes of nilpotents in* $\mathfrak{p}_{\mathrm{C}}$*, the regular class* $K_{\mathrm{C}} \cdot e_1$ *and the non-regular (minimal) class* $K_{\mathrm{C}} \cdot e_2$. e_1 *and* e_2 *may be chosen to lie in the same* θ*-stable Borel subalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}_{\mathrm{C}}$. $A_k(e_1) = \{1\}$ *and we associate the pair* $(e_1,\ \{1\})$ *to the triple* $(sl(3,\ \mathbf{C}),\ \mathfrak{b},\ \mathbf{C}e_1)$. $A_k(e_2) = \mathbf{Z}_2 = \{1,\ -1\}$. *We associate* $(e_2,\ \{1\})$ *to the triple* $(sl(3,\ \mathbf{C}),\ \mathfrak{b},\ \mathbf{C}e_2)$*, and we associate* $(e_2,\ \{-1\})$ *to the triple* $(sl(2,\ \mathbf{R}) \oplus V,\ \mathfrak{b}',\ \mathbf{C}e_2)$. *Here* V *is a one dimensional real subspace of* $\mathfrak{p}$ *and* $\mathfrak{b}'$ *is a Borel subalgebra in* $sl(2,\ \mathbf{C}) \oplus V_{\mathbf{C}}$.

Our second example concerns $A_k(e)$ for conjugacy class 23 of EI, the split real form of the complex simple algebra E_6. (See [2] and Table 4 of [6])

Example 2. *Let* e *be a representative of class 23 of* $\mathfrak{g} = EI$. $A_k(e) = S_4$ *has 5 conjugacy classes. Since* e *is noticed in* $\mathfrak{g}$*, the pair* $(e,\ id)$ *is associated to* $(EI,\ \mathfrak{q},\ W)$. *There are two conjugacy classes of elements of order 2. These conjugacy classes are associated to triples* $(so(5,\ 5) \oplus V',\ \mathfrak{q}',\ W')$ *and* $(sl(6,\ \mathbf{R}) \oplus sl(2,\ \mathbf{R}),\ \mathfrak{q}'',\ W'')$. *The conjugacy class of elements of order 3 is associated to the triple* $(su(2,\ 1) \oplus sl(3,\ \mathbf{C}),\ \mathfrak{q}''',\ W''')$. *The conjugacy class of elements of order 4 is associated to the triple* $(su(2,\ 2) \oplus sl(2,\ \mathbf{C}) \oplus V'',\ \mathfrak{q}^{iv},\ W^{iv})$. V'*, and* V'' *are one dimensional abelian vector subspaces of* $\mathfrak{p}$.

In general, Theorem 1 provides an efficient way of organizing the calculation of $A_k(e)$ when $\mathfrak{g}$ is a simple exceptional algebra. The theorem often simplifies the work done in [6].

In order to state our second result, we will define the notion of the r-carrier subalgebra of a nilpotent e in $\mathfrak{p}_{\mathbf{C}}$ (see Vinberg's notion of "carrier subalgebra" in [15]). This algebra will be denoted $\mathfrak{s} = \mathfrak{s}(e)$ and will be defined as a $\mathbf{Z}$ graded algebra.

Assume that $\{x,\ e,\ f\}$ is a θ-stable, σ-stable sl(2) triple. By [12] each nilpotent in $\mathfrak{p}_{\mathrm{C}}$ is K_{C} conjugate to an element e lying in such a triple. Set $\mathfrak{g}_{\mathrm{C}}(i)$ equal to the i eigenspace of $ad(x)$ on $\mathfrak{g}_{\mathbf{C}}$. If i is odd, set $\mathfrak{s}_i = \mathfrak{g}_{\mathrm{C}}(2i) \cap \mathfrak{p}_{\mathrm{C}}$. If i is even, set $\mathfrak{s}_i = \mathfrak{g}_{\mathrm{C}}(2i) \cap \mathfrak{k}_{\mathrm{C}}$. Then,

$$\mathfrak{s} = \mathfrak{s}(e) \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbf{Z}} \mathfrak{s}_i$$

It is known that $\mathfrak{s}$ is reductive [16]. We will call it the r-carrier subalgebra of e in $\mathfrak{g}_{\mathbf{C}}$. (The algebra $\mathfrak{s}$ also occurs in the work of Kawanaka [5].) Let $\widetilde{\mathfrak{g}}_{\mathbf{C}}(e) = \widetilde{\mathfrak{g}}_{\mathbf{C}}(\{x,\ e,\ f\})$ denote the semisimple part of $\mathfrak{s}$. It is the carrier subalgebra (in the sense of Vinberg) of e in $\mathfrak{g}_{\mathbf{C}}$. The connected subgroup of $G_{\mathbf{C}}$ with Lie algebra $\widetilde{\mathfrak{g}}_{\mathbf{C}}(e)$ is denoted $\widetilde{G}_{\mathbf{C}}(e)$ and is called the carrier subgroup of e. e is distinguished in $\widetilde{\mathfrak{g}}_{\mathbf{C}}(e)$, i.e, $\widetilde{\mathfrak{g}}_{\mathbf{C}}(e)^{\{x,\ e,\ f\}} = (0)$ by Theorem 2 of [16]. $\widetilde{\mathfrak{g}}_{\mathbf{C}}(e)$ is also locally flat in Vinberg's terminology [15].

Clearly $\widetilde{\mathfrak{g}}_{\mathbf{C}}(e)$ is τ stable so that it is the complexification of a subalgebra $\widetilde{\mathfrak{g}}(e)$ of $\mathfrak{g}$. $\widetilde{\mathfrak{g}}(e)$ is of equal rank since one can find a Cartan subalgebra of $\widetilde{\mathfrak{g}}(e)$ containing x inside $\mathfrak{s}_0$. Note that if e is noticed, then $\mathfrak{s}(e) = \widetilde{\mathfrak{g}}_{\mathbf{C}}(e)$. The reason is that any element z of the center of $\mathfrak{s}$ must belong to $\mathfrak{s}_0$ and hence to $\mathfrak{k}_{\mathbf{C}}$. But since e is noticed, $\mathfrak{k}_{\mathbf{C}}^{\{x,\ e,\ f\}} = 0$. So $z = 0$.

Theorem 2. *With notation as above. Assume that* e *is noticed in* $\mathfrak{g}_{\mathbf{C}}$ *and* $\widetilde{G_{\mathrm{C}}}(e)$ *is simple and the adjoint group of* $\widetilde{\mathfrak{g}}_{\mathbf{C}}(e)$. *Then, there is a bijection from* $K_{\mathbf{C}}$ *conjugacy classes of triples* $(\mathfrak{l},\ \mathfrak{q}_l,\ W)$ *in Theorem 1 (corresponding to* e*) to* $\widetilde{G}_{\mathbf{C}}(e)$ *conjugacy classes of pairs* $(\widetilde{\mathfrak{l}},\ \mathfrak{q}_{\widetilde{l}})$ *where* $\widetilde{\mathfrak{l}}$ *is a pseudo Levi subalgbra of* $\widetilde{\mathfrak{g}}_{\mathbf{C}}(e)$ *and*

$\mathfrak{q}_{\tilde{l}}$ is a distinguished parabolic of $\tilde{\mathfrak{l}}$ containing e. In particular, $A_k(e)$ is isomorphic to the component group of e in $\widetilde{G_{\mathbf{C}}}(e)$. In this correspondence, $\tilde{\mathfrak{l}}$ is the r-carrier subalgebra of e in $\mathfrak{l}$.

Theorem 2 shows that in some sense Theorem 1 is already contained in Theorem 14 of [14].

Here are two examples of illustrating Theorem 2.

Example 3. *Consider $K_{\mathbf{C}} \cdot e$, the unique $K_{\mathbf{C}}$ conjugacy class for the split form of G_2 for which $A_k(e) = S_3$. The carrier subalgebra of e is G_2 itself. The carrier subgroup is the adjoint group of G_2. e is distinguished in $\mathfrak{g}_{\mathbf{C}} = G_2$ and in each of the pseudo-Levi subalgebras occurring in a triple of the type considered in Theorem 1. Thus, the first two elements of each of these triples constitute a Sommers pair. The correspondence asserted in Theorem 2 is trivial in this case.*

Example 4. *Our second example for Theorem 2 concerns the conjugacy class treated in Example 2. Table I relates the triples for class 23 of EI (from Example 2) to the Sommers pairs for the class $F_4(a_3)$ in the complex simple algebra F_4. (See [14]). We give only the semisimple part of the real form of each elliptic pseudo Levi subalgebra of EI.*

Table I

pseudo − Levi subalgebra	*carrier subalgebra*
EI	$F4(a_3)$
$so(5,\ 5)$	$B_4(a_1)$
$sl(6,\ \mathbf{R}) \oplus sl(2,\ \mathbf{R})$	$A_1 \oplus C_3(a_1)$
$su(2,\ 1) \oplus sl(3,\ \mathbf{C})$	$A_2 \oplus \widetilde{A_2}$
$su(2,\ 2) \oplus sl(2,\ \mathbf{C})$	$A_3 \oplus \widetilde{A_1}$

2. Discussion and Proofs

Recall that $\mathfrak{g}$ is simple. Let $\mathfrak{t}$ be a maximal torus of $\mathfrak{k}$. $\mathfrak{h} = \mathfrak{g}^{\mathfrak{t}}$ is the fundamental Cartan subalgebra of $\mathfrak{g}$ which contains $\mathfrak{t}$. Set $\tilde{\mathfrak{a}} = \mathfrak{h} \cap \mathfrak{p}$. The complexifications of $\mathfrak{t}$ and $\mathfrak{h}$ will be denoted $\mathfrak{t}_{\mathbf{C}}$ and $\mathfrak{h}_{\mathbf{C}}$ respectively. $\mathfrak{t}^* = Hom_{\mathbf{R}}(\mathfrak{t},\ \mathbf{R})$ and $\mathfrak{h}^*_{\mathbf{C}} = Hom_{\mathbf{C}}(\mathfrak{h}_{\mathbf{C}},\ \mathbf{C})$. $H_{\mathbf{C}}$ (resp., $T_{\mathbf{C}}$) is the connected subgroup of $G_{\mathbf{C}}$ (resp., $K_{\mathbf{C}}$) with Lie algebra $\mathfrak{h}_{\mathbf{C}}$ (resp., $\mathfrak{t}_{\mathbf{C}}$). Let Δ denote the set of roots of the pair $(\mathfrak{g}_{\mathbf{C}},\ \mathfrak{h}_{\mathbf{C}})$. If $\beta \in \Delta$, let e^{β} denote the corresponding character of $H_{\mathbf{C}}$.

If π is a system of simple roots for a positive system $\Delta^+ = \Delta^+(\pi)$ of Δ, let ψ denote the highest root of Δ^+. From now on, assume that π is θ-stable. Then ψ is imaginary. Set $\tilde{\pi} = \pi \cup \{-\psi\}$.

If $J \subsetneq \tilde{\pi}$, let Δ_J be the root system spanned by J. Define $(\mathfrak{g}_{\mathbf{C}})_J$ to be the subalgebra of $\mathfrak{g}_{\mathbf{C}}$ spanned by $\mathfrak{h}_{\mathbf{C}}$ and the root vectors X_{ϕ} with $\phi \in \Delta_J$. (This agrees with the definition of $(\mathfrak{g}_{\mathbf{C}})_J$ in [14].) From now on we will assume that J is θ stable. Let $(G_{\mathbf{C}})_J$ denote the connected subgroup of $G_{\mathbf{C}}$ with Lie algebra $(\mathfrak{g}_{\mathbf{C}})_J$. $(\mathfrak{g}_{\mathbf{C}})_J$ (resp., $(G_{\mathbf{C}})_J$) is said to be a **standard elliptic pseudo Levi subalgebra** (resp., **standard elliptic pseudo Levi subgroup**) of $\mathfrak{g}_{\mathbf{C}}$ (resp. $G_{\mathbf{C}}$). $(K_{\mathbf{C}})_J$

denotes the connected subgroup of $(G_{\mathbb{C}})_J$ with Lie algebra $(\mathfrak{g}_{\mathbb{C}})_J \cap \mathfrak{k}_{\mathbb{C}}$. The center of $(G_{\mathbb{C}})_J$ will be denoted by $Z((G_{\mathbb{C}})_J)$ or by Z_J.

Since $(\mathfrak{g}_{\mathbb{C}})_J$ is stable under σ, $(\mathfrak{g}_{\mathbb{C}})_J \cap \mathfrak{g}$ is a real form of $(\mathfrak{g}_{\mathbb{C}})_J$. Set $\mathfrak{g}_J = (\mathfrak{g}_{\mathbb{C}})_J \cap \mathfrak{g}$, and G_J equal to the connected subgroup of G with Lie algebra $\mathfrak{g}_J$. We will often describe $(\mathfrak{g}_{\mathbb{C}})_J$ by specifying $\mathfrak{g}_J$.

$\mathbf{Z}[\Delta_k]$ is the $\mathbf{Z}$ lattice spanned by Δ_k, the roots of $(\mathfrak{k}_{\mathbf{C}},\ \mathfrak{t}_{\mathbf{C}})$. $\mathbf{L}$ is the root lattice spanned by π. $\mathbf{L}_J \subset \mathbf{L}$ is the root lattice spanned by J. $\mathbf{L}_J(\mathfrak{t})$ is the $\mathbf{Z}$ lattice spanned by the restrictions of the roots of Δ_J to $\mathfrak{t}_{\mathbf{C}}$. $\mathbf{W}$ is the Weyl group of Δ. $\mathbf{W}_J$ is the Weyl group of Δ_J and $\mathbf{W}_k$ is the Weyl group of Δ_k.

$Z_J \cap K_{\mathbf{C}} \subset T_{\mathbf{C}} = \exp(\mathfrak{t}_{\mathbf{C}})$. Set $\widetilde{(\mathfrak{t}_{\mathbf{C}})}_J$ (resp., $\widetilde{(\mathfrak{h}_{\mathbf{C}})}_J$) equal to the complex subspace of $\mathfrak{t}_{\mathbf{C}}$ (resp., $\mathfrak{h}_{\mathbf{C}}$) corresponding to $\mathbf{L}_J(\mathfrak{t})$ (resp., $\mathbf{L}_J$), and $\mathfrak{t}_{\mathbf{C}}^{\mathbf{L}_J}$ (resp., $\mathfrak{h}_{\mathbf{C}}^{\mathbf{L}_J}$) equal to the subspace of $\mathfrak{t}_{\mathbf{C}}$ (resp., $\mathfrak{h}_{\mathbf{C}}$) where all $\gamma \in \mathbf{L}_J$ vanish. $T_{\mathbf{C}}^{\mathbf{L}_J}$ (resp., $H_{\mathbf{C}}^{\mathbf{L}_J}$) denotes the elements of $T_{\mathbf{C}}$ (resp., $H_{\mathbf{C}}$) where the elements of $\mathbf{L}_J(\mathfrak{t})$ (resp., $\mathbf{L}_J$) viewed as group characters are trivial. Then $\mathfrak{t}_{\mathbf{C}}^{\mathbf{L}_J}$ (resp., $\mathfrak{h}_{\mathbf{C}}^{\mathbf{L}_J}$) is the Lie algebra of $T_{\mathbf{C}}^{\mathbf{L}_J}$ (resp., $H_{\mathbf{C}}^{\mathbf{L}_J}$). Moreover,

$$\begin{aligned} \mathfrak{t}_{\mathbf{C}} &= \widetilde{(\mathfrak{t}_{\mathbf{C}})}_J \oplus \mathfrak{t}_{\mathbf{C}}^{\mathbf{L}_J} \\ Z_J &= H_{\mathbf{C}}^{\mathbf{L}_J} \\ Z_J \cap K_{\mathbf{C}} &= T_{\mathbf{C}}^{\mathbf{L}_J} \\ (Z_J \cap K_{\mathbb{C}})_0 &= \exp(\mathfrak{t}_{\mathbf{C}}^{\mathbf{L}_J}) \end{aligned}$$

Definition 5. *Suppose that* $\psi = \sum_{\alpha\in\pi} c_\alpha \alpha$. *Set* d_J *equal to the greatest common divisor of the* c_α *such that* $\alpha \in \widetilde{\pi} - J$.

Recall Sommers'decomposition of Z_J as $(Z_J)_0 \times \mathbf{Z}_{d_J}$, where $\mathbf{Z}_{d_J}$ is the cyclic group of order d_J. Sommers shows that $Z_J/(Z_J)_0$ is isomorphic to $tor(\mathbf{L}/\mathbf{L}_J)$, the torsion subgroup of $\mathbf{L}/\mathbf{L}_J$. In addition, recall the following element defined by Sommers:

$$\tau_J = (-1/d_J) \sum_{\alpha\in\tilde{\pi}-J} c_\alpha \alpha. \tag{2.1}$$

The coset $\tau_J + \mathbf{L}_J$ can be taken as the generator of $tor(\mathbf{L}/\mathbf{L}_J) \simeq \mathbf{Z}_{d_J}$. Since J is θ-stable, if we view τ_J as an element of $\mathfrak{h}_{\mathbb{C}}^*$, it is fixed by θ and so belongs to $\mathfrak{t}_{\mathbb{C}}^*$. For each $\alpha \in \pi$, let ω_α denote the element of $\mathfrak{h}_{\mathbb{C}}$ which corresponds to the fundamental weight for α. Thus, for each $\beta \in \pi$, $\dfrac{(2\omega_\alpha,\ \beta)}{(\alpha,\ \alpha)} = 0$ if $\beta \neq \alpha$, and $= 1$, if $\beta = \alpha$.

Sommers shows that if $d_J = 1$, then $Z_J = (Z_J)_0$, i.e., Z_J is connected. If $d_J > 1$, then $-\psi \in J$ and we have.

Lemma 1. *Assume that* $d_J > 1$. *For each non-negative integer* n, *set*

$$h_n = \frac{-2\pi i}{d_J^n} \sum_{\alpha\in\tilde{\pi}-J} \frac{2\omega_\alpha}{(\alpha,\ \alpha)}.$$

Then: (a) For some unique N, $\exp(h_N) \in (Z_J)_0$, *and* $\exp(h_{N+1}) \in Z_J \setminus (Z_J)_0$. *(b) Choose* Q *to be a positive integer such that* $\dfrac{\sum_{\alpha\in\tilde{\pi}-J} c_\alpha}{Q d_J^{N+1}}$ *and* d_J *are relatively* *prime. If we set* $\widetilde{\tau_J} = \exp(\frac{h_{N+1}}{Q})$*, then* $\widetilde{\tau_J}$ *is a representative of the cyclic generator of* $\frac{Z_J}{(Z_J)_0}$.

Proof. (a) First, consider $\exp(h_1)$. It clearly belongs to Z_J. Notice that $\exp(h_0) = \exp(d_J h_1) = e$ which belongs to $(Z_J)_0$. If $\exp(h_1)$ does not belong to $(Z_J)_0$, we are done. If it belongs to $(Z_J)_0$, there is an element $h \in \mathfrak{h}_{\mathbf{C}}^{\mathbf{L}_J}$ such that $\exp(h_1) = \exp(h)$. This means that $h = \sum_{\alpha\in\tilde{\pi}-J} d_\alpha \frac{2\omega_\alpha}{(\alpha,\ \alpha)}$ with each $d_\alpha \in \mathbf{C}$, and $\sum_{\alpha\in\tilde{\pi}-J} c_\alpha d_\alpha = 0$. Since $\exp(h_1) = \exp(h)$, $\beta(h - h_1) \in 2\pi i\mathbf{Z}$ for all $\beta \in \Delta$.

It follows that for each $\alpha \in \widetilde{\pi} - J$, there is an integer k_α such that:

$$\frac{-2\pi i}{d_J} - 2\pi i k_\alpha = d_\alpha \tag{2.2}$$

Since $\sum_{\alpha\in\tilde{\pi}-J} c_\alpha d_\alpha = 0$, (2.2) implies that:

$$-2\pi i\frac{\sum_{\alpha\in\tilde{\pi}-J} c_\alpha}{d_J} = 2\pi i \sum_{\alpha\in\tilde{\pi}-J} c_\alpha k_\alpha \tag{2.3}$$

and hence,

$$\frac{-\sum_{\alpha\in\tilde{\pi}-J} c_\alpha}{d_J} = \sum_{\alpha\in\tilde{\pi}-J} c_\alpha k_\alpha \tag{2.4}$$

Since d_J divides each c_α on the the right hand side of (2.4), d_J must also divide the left side. This implies that $\exp(h_2) \in Z_J$ (since ψ and all the roots in $\pi \cap J$ are trivial on $\exp(h_2)$). Continuing in this fashion we arrive at an N for which $\exp(h_N) \in (Z_J)_0$ but $\exp(h_{N+1}) \in Z_J \setminus (Z_J)_0$. For this value of N, we must have that $\frac{\sum_{\alpha\in\tilde{\pi}-J} c_\alpha}{d_J^{N+1}}$ is not divisible by d_J. Otherwise, $\exp(h_{N+2}) \in Z_J$, but $\exp(h_{N+1}) = \exp(h_{N+2})^{d_J} \notin (Z_J)_0$ which contradicts the fact that $\frac{Z_J}{(Z_J)_0}$ is cyclic of order d_J. This establishes the assertion of part (a).

Note that since $\tilde{\pi}$ and J are θ-stable, $\tilde{\pi} - J$ is θ-stable, so each h_n is fixed by θ and hence belongs to $\mathfrak{t}$.

(b) is proven like (a). First note that $\widetilde{\tau_J} \notin (Z_J)_0$, since otherwise $\exp(h_{N+1}) \in (Z_J)_0$. Moreover, one can show that if $\widetilde{\tau_J}^m = \exp(m(\frac{h_{N+1}}{Q})) \in (Z_J)_0$, then $\frac{m\sum_{\alpha\in\tilde{\pi}-J} c_\alpha}{Qd_J^{N+1}}$ is divisible by d_J. But this implies that d_J divides m. On the other hand, it is clear that $\widetilde{\tau_J}^{d_J} \in (Z_J)_0$.

Let $\mathfrak{l}$ be a standard elliptic pseudo Levi subalgebra. Let L be the connected subgroup in $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{l}$. We have the exact sequence:

$$(1) \to \frac{Z(L)_0 \cap K_{\mathbf{C}}}{(Z(L) \cap K_{\mathbf{C}})_0} \to \frac{Z(L) \cap K_{\mathbf{C}}}{(Z(L) \cap K_{\mathbf{C}})_0} \to \frac{Z(L) \cap K_{\mathbf{C}}}{Z(L)_0 \cap K_{\mathbf{C}}} \to (1). \tag{2.5}$$

Let $z(\mathfrak{l})$ be the center of $\mathfrak{l}$. Then, $Z(L)_o = \exp(z(\mathfrak{l}) \cap \mathfrak{k}_{\mathbf{C}}) \exp(z(\mathfrak{l}) \cap \mathfrak{p}_{\mathbf{C}})$.

Recall that $\widetilde{\mathfrak{u}} = \mathfrak{k} + i\mathfrak{p}$. Set $\widetilde{\mathfrak{u}}_L = \mathfrak{l} \cap \mathfrak{k} + i(\mathfrak{l} \cap \mathfrak{p})$. Let the corresponding connected subgroups of $G_{\mathbf{C}}$ be denoted $\widetilde{U}$ and $\widetilde{U_L}$ respectively. $\widetilde{U}$ and $\widetilde{U_L}$ are maximal compact subgroups of $G_{\mathbf{C}}$ and L respectively.

Proposition 1. *Suppose that $\mathfrak{l}$ is a standard elliptic pseudo Levi subgroup. Then*
(1) $Z(L) \cap K_{\mathbf{C}} \,/\, (Z(L)_0 \cap K_{\mathbf{C}}) \simeq Z(L)/Z(L)_0$
(2) $Z(L)_0 \cap K_{\mathbf{C}}/(Z(L) \cap K_{\mathbf{C}})_0$ *is a 2-group.*

Proof. Lemma 1 implies that there is an element $z \in Z(L) \cap K_{\mathbf{C}}$ such that the coset $zZ(L)_0$ generates $Z(L)/Z(L)_0$. (1) follows. In addition, we have $Z(L) = Z(L) \cap K_{\mathbf{C}} \exp(z(\mathfrak{l}) \cap \mathfrak{p}_{\mathbf{C}}))$.

Also, $Z(L)_0 = \exp(z(\mathfrak{l})) = \exp(z(\mathfrak{l}) \cap \mathfrak{k}_{\mathbf{C}}) \exp(z(\mathfrak{l}) \cap \mathfrak{p}_{\mathbf{C}})$. Since $(Z(L) \cap K_{\mathbf{C}})_0 = \exp(z(\mathfrak{l}) \cap \mathfrak{k}_{\mathbf{C}})$, there is a surjective homomorphism:

$$\exp(z(\mathfrak{l}) \cap \mathfrak{p}_{\mathbf{C}}) \cap K_{\mathbf{C}} \to \frac{Z(L)_0 \cap K_{\mathbf{C}}}{(Z(L) \cap K_{\mathbf{C}})_0}$$

A simple argument invoking the diffeomorphism $\widetilde{U}_L \times i\widetilde{\mathfrak{u}}_L \to L$ (given by $(h,\, Y) \mapsto h\exp(Y)$) shows that each non-trivial element of $\exp(z(\mathfrak{l}) \cap \mathfrak{p}_{\mathbf{C}}) \cap K_{\mathbf{C}}$ has order two and belongs to the subgroup $\exp(i(z(\mathfrak{l}) \cap \mathfrak{p})) \cap K$. $z(\mathfrak{l}) \cap \mathfrak{p}$ can be extended to a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{l}_{\mathbf{R}} \cap \mathfrak{p}$. It is known that the subgroup of the group $\exp(i\mathfrak{a})$ consisting of all elements of order two is finite with $\dim_{\mathbf{R}} \mathfrak{a}$ generators. This establishes (2).

We can characterize the standard elliptic Levi subalgebras among the standard elliptic pseudo Levi subalgebras in terms of the corresponding θ stable subset of $\widetilde{\pi}$. We leave the proof of the following result to the reader.

Lemma 2. $(\mathfrak{g}_{\mathbf{C}})_J$ *is an elliptic Levi subalgebra if and only if* $-\psi \notin J$.

Proposition 2. *Let J and J' be two proper θ-stable subsets of $\widetilde{\pi}$ where π is a θ-stable simple system of Δ. Consider the following statements:*
(1) J and J' are conjugate under $\mathbf{W}_k$.
(2) Δ_J and $\Delta_{J'}$ are conjugate under $\mathbf{W}_k$.
(3) $\mathbf{W}_J$ and $\mathbf{W}_{J'}$ are conjugate under $\mathbf{W}_k$.
(4) $(\mathfrak{g}_{\mathbf{C}})_J$ and $(\mathfrak{g}_{\mathbf{C}})_{J'}$ are conjugate under $K_{\mathbf{C}}$.
Then, (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4).

Proof. (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) is clear. (2) $\Longrightarrow$ (4) also follows because if $w(\Delta_J) = \Delta_{J'}$ for $w \in \mathbf{W}_k$, then any element $k \in N_K(T)$ which represents w will conjugate $(\mathfrak{g}_{\mathbf{C}})_J$ to $(\mathfrak{g}_{\mathbf{C}})_{J'}$. Conversely, (4) $\Longrightarrow$ (2), because if $k \in K_{\mathbf{C}}$ and $k \cdot (\mathfrak{g}_{\mathbf{C}})_J = (\mathfrak{g}_{\mathbf{C}})_{J'}$, then $k \cdot \mathfrak{h}_{\mathbf{C}}$ will be the complexification of a fundamental Cartan subalgebra of $\mathfrak{g}$. We may as well assume that $k \cdot \mathfrak{t}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}}$, and so k corresponds to an element of $\mathbf{W}_k$. It is clear that k sends Δ_J to $\Delta_{J'}$.

Remark 3. *Let $\mathfrak{g} = su(2,\, 1)$ Assume that $\pi = \{\alpha,\, \beta\}$ with α compact and β non-compact. Then $\psi = \alpha + \beta$ is non-compact. $\widetilde{\pi} = \{\alpha,\, \beta,\, -\psi\}$. Let $J = \{\alpha,\, \beta\}$ and $J' = \{\beta,\, -\psi\}$. Then J and J' are not conjugate under $\mathbf{W}_k$ but $(\mathfrak{g}_{\mathbf{C}})_J = (\mathfrak{g}_{\mathbf{C}})_{J'}$. Thus, (2), (3) and (4) of the previous proposition do not imply (1).*

Fix π to be a θ stable set of simple roots for the fundamental Cartan subalgebra of $\mathfrak{g}$. Let J be a proper θ stable subset of $\widetilde{\pi}$. If Γ is an abelian group, let $tor(\Gamma)$ denote the torsion subgroup of Γ.

Proposition 3. *Assume that $\mathfrak{g}$ is simple.*

(a) If $\mathfrak{g}$ is equal rank, then $\dfrac{Z_J \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$ *is isomorphic to* $tor(\dfrac{\mathbf{L}}{\mathbf{L}_J})$.

(b) If $\mathfrak{g}$ is not equal rank, then $\dfrac{Z_J \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$ *is isomorphic to* $tor\left(\dfrac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$.

Proof. Let $X^*(T_{\mathbf{C}})$ denote the set of characters of $T_{\mathbf{C}}$. In [14], Sommers proves that $\dfrac{Z_J}{(Z_J)_0}$ is isomorphic to $tor(\dfrac{\mathbf{L}}{\mathbf{L}_J})$ by analyzing $Z_J = H_{\mathbf{C}}^{\mathbf{L}_J}$. Since $Z_J \cap K_{\mathbf{C}} = T_{\mathbf{C}}^{\mathbf{L}_J(\mathfrak{t})}$, we can copy Sommers argument and deduce the following lemma.

Lemma 3. $\dfrac{Z_J \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$ *is isomorphic to* $tor\left(\dfrac{X^*(T_{\mathbf{C}})}{\mathbf{L}_J(\mathfrak{t})}\right)$.

We now prove part (a) of Proposition 3. Since $\mathfrak{g}$ is equal rank, $Z_J \subset T_{\mathbf{C}} \subset K_{\mathbf{C}}$ which implies that $\dfrac{Z_J \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0} = \dfrac{Z_J}{(Z_J)_0}$. So the desired conclusion follows from Sommers result cited before the previous lemma.
We now prove part (b) of Proposition 3. Since $\mathfrak{g}$ is not equal rank, one verifies case by case that $K_{\mathbf{C}}$ is equal to the adjoint group of $\mathfrak{k}_{\mathbf{C}}$. Therefore, $X^*(T_{\mathbf{C}}) = \mathbf{Z}[\Delta_k]$. The desired conclusion follws from the previous lemma.

There is considerable evidence that the previous proposition can be expressed more succinctly as follows.

Conjecture 1. *If $\mathfrak{g}$ is simple, and J is as above,* $\dfrac{Z_J \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$ *is isomorphic to* $tor\left(\dfrac{\mathbf{Z}[\Delta_k]}{\mathbf{Z}[\Delta_k] \cap \mathbf{L}_J(\mathfrak{t})}\right)$.

Proposition 4. *Suppose that $\mathfrak{g}$ is simple and not equal rank.*

(a) If $-\psi$ does not belong to J then the 2 subgroup of $tor\left(\dfrac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ *is trivial, and hence* $\dfrac{(Z_J)_0 \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$ *is trivial.*

(b) Suppose that $-\psi$ belongs to J. Then, the following are equivalent (i) $\tau_J/2$ belongs to $\mathbf{Z}[\Delta_k]$; (ii) $\dfrac{(Z_J)_0 \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$ *is equal to $\mathbf{Z}_2$; (iii)* $\dfrac{(Z_J)_0 \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$ *is non trivial.*

(c) If $-\psi$ belongs to J and $\dfrac{(Z_J)_0 \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$ *is non trivial, then for each $\alpha \in (\widetilde{\pi} - J)_c$ or $(\widetilde{\pi} - J)_{nc}$, c_α is even.*

(d) Assume $-\psi$ belongs to J. If $\tau_J/2$ belongs to $\mathbf{Z}[\Delta_k]$, then $tor\left(\dfrac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ *is cyclic of order $2d_J$. If $\tau_J/2 \notin \mathbf{Z}[\Delta_k]$, then $(Z_J)_0 \cap K_{\mathbf{C}}$ is connected and* $tor\left(\dfrac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ *is cyclic of order d_J .*

Proof.

Since $\mathfrak{g}$ is not equal rank the restriction of each root in Δ to $\mathfrak{t}_{\mathbf{C}}$ belongs to $\mathbf{Z}[\Delta_k]$. Moreover, ψ is imaginary and so may be identified with its restriction to $\mathfrak{t}_{\mathbf{C}}$. If ϕ is a complex linear functional on $\mathfrak{h}_{\mathbf{C}}$, $\phi^\sharp$ will denote its restriction to $\mathfrak{t}_{\mathbf{C}}$.

Let π_c (resp., J_c), π_{nc} (resp., J_{nc}), and π_{cx} (resp.. J_{cx}), denote respectively the subsets of compact imaginary, noncompact imaginary and complex roots in π (resp., J.). Δ_c, Δ_{nc}, and Δ_{cx} have the obvious meaning. If $\phi \in \Delta_{cx}$, then its restriction $\phi^\sharp \in \mathbf{Z}[\Delta_k]$ and $\phi^\sharp = (\phi + \theta(\phi))/2$. Every root in Δ_k is either a compact imaginary root or the restriction to $\mathfrak{t}_{\mathbf{C}}$ of a complex root of $\mathfrak{g}$. Moreover, since $\mathfrak{g}$ is not equal rank, each noncompact imaginary root belongs to $\mathbf{Z}[\Delta_k]$.

(a) Since $-\psi$ does not belong to J, $d_J = 1$ and Z_J is connected. Hence , $\dfrac{Z_J \cap K_{\mathbf{C}}}{(Z_J)_0 \cap K_{\mathbf{C}}}$ is trivial. Therefore, by Proposition 1, $\dfrac{Z_J \cap K_{\mathbf{C}}}{(Z_J)_0 \cap K_{\mathbf{C}}} = \dfrac{(Z_J)_0 \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$ is a 2 group.

To complete the proof of (a) we need the following lemma.

Lemma 4. *Suppose that $\lambda \in \mathbf{L}_J(\mathfrak{t})$ and $\lambda/2 \in \mathbf{Z}[\Delta_k]$, then $\lambda/2 \in \mathbf{L}_J(\mathfrak{t})$.*

Proof of lemma. On the one hand, for some integers $n_\alpha \in \mathbf{Z}$,

$$\lambda = \sum_{\alpha \in J_{nc}} n_\alpha \alpha + \sum_{\alpha \in J_c} n_\alpha \alpha + \sum_{\{\alpha, \theta(\alpha)\}, \alpha \in J_{cx}} n_\alpha \alpha^\sharp. \tag{2.6}$$

On the other hand for some integers $m_\phi \in \mathbf{Z}$, we have

$$\lambda/2 = \sum_{\phi \in \Delta_c} m_\phi \phi + \sum_{\{\phi, \theta(\phi)\}, \phi \in \Delta_{cx}} m_\phi \phi^\sharp,$$

or,

$$\lambda = \sum_{\phi \in \Delta_c} 2m_\phi \phi + \sum_{\{\phi, \theta(\phi)\}, \phi \in \Delta_{cx}} 2m_\phi \phi^\sharp, \tag{2.7}$$

Each $\phi \in \Delta_c$ is an integral combination of imaginary and complex roots in π. Each $\phi^\sharp$ with $\phi \in \Delta_{cx}$ is a sum of an integral combination of imaginary roots in π and an integral combination of restrictions of complex roots in π. Thus 2.7 implies that we can write:

$$\lambda = \sum_{\alpha \in \pi_{nc}} 2p_\alpha \alpha + \sum_{\alpha \in \pi_c} 2p_\alpha \alpha + \sum_{\{\alpha, \theta(\alpha)\}, \alpha \in \pi_{cx}} 2p_\alpha \alpha^\sharp. \tag{2.8}$$

where $p_\alpha \in \mathbf{Z}$. Since J and π are θ stable systems of simple roots and the roots in each set are linearly independent, when we equate 2.6 and 2.8, we conclude that n_α is even (possibly equal to zero) for each $\alpha \in J$. It follows that $\lambda/2 \in \mathbf{L}_J(\mathfrak{t})$, completing the proof of the lemma.

Lemma 4 implies that there are no non trivial elements of order two in $\dfrac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}$. Since $\dfrac{(Z_J)_0 \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$ is a 2 subgroup, it must be trivial.

(b) Clearly (ii) $\Longrightarrow$ (iii).

We now show that (iii) $\Longrightarrow$ (ii). Part (c) will be a corollary of this argument. So assume (iii) holds. Then the 2 subgroup of $tor\left(\dfrac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ is not trivial. Recalling that $\psi = \sum_{\alpha \in \pi} c_\alpha \alpha$, set

$$\psi_{\pi \cap J} = \sum_{\alpha \in \pi \cap J} c_\alpha \alpha \quad \text{and} \quad \psi_{\tilde{\pi} - J} = \sum_{\alpha \in \tilde{\pi} - J} c_\alpha \alpha.$$

The following argument shows that every non trivial element of the 2 subgroup is represented by $\psi_{\widetilde{\pi}-J}/2$.

Any element of the 2 subgroup of $tor\left(\frac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ has a representative in $\mathbf{Z}[\Delta_k]$ of the form $n(-\psi/2)+\lambda'/2$ where $n\in\mathbf{Z}$ and $\lambda'=\sum_{\alpha\in\pi\cap J} n_\alpha\alpha^\sharp$. (In the previous sum, each complex simple root is counted only once.) Since $\psi\in\mathbf{L}_J(\mathfrak{t})$, we may assume that $n=0$ or 1. If $n=0$, then Lemma 4 implies that $\lambda'/2\in\mathbf{L}_J(\mathfrak{t})$. So if the representative is nontrivial, we may assume that $n=1$.

If we argue as in part (a), we conclude that: (1) if $\alpha\in(\pi\cap J)$ is imaginary, then $n_\alpha-c_\alpha$ is even; (2) if $\alpha\in(\pi\cap J)_{cx}$, then $n_\alpha-2c_\alpha$ is even; and (3) if $\alpha\in(\widetilde{\pi}-J)$ is imaginary, then c_α is even. Therefore, $-\psi_{\pi\cap J}/2+\lambda'/2$ lies in $\mathbf{L}_J(\mathfrak{t})$ and $-\psi_{\widetilde{\pi}-J}/2$ belongs to $\mathbf{Z}[\Delta_k]$.

One can show using the linear independence of the simple roots that $-\psi_{\widetilde{\pi}-J}/2$ cannot belong to $\mathbf{L}_J(\mathfrak{t})$. Thus, if the 2 subgroup of $tor\left(\frac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ is non-trivial it is generated by $-\psi_{\widetilde{\pi}-J}/2+\mathbf{L}_J(\mathfrak{t})$. We have established (iii) $\Longrightarrow$ (ii).

Again using the linear independence of the roots in π, one can show that if $-\psi_{\widetilde{\pi}-J}/2$ lies in $\mathbf{Z}[\Delta_k]$, then for each $\alpha\in(\widetilde{\pi}-J)_c$ or $(\widetilde{\pi}-J)_{nc}$, c_α is even. This is part (c).

We now show that (i) $\Longrightarrow$ (iii). Assuming (i) holds, we construct an element of $(Z_J)_0\cap K_{\mathbf{C}}$ which does not lie in $(Z_J\cap K_{\mathbf{C}})_0$. The key observation is that (i) implies that if $\alpha\in(\widetilde{\pi}-J)_c$ or $\alpha\in(\widetilde{\pi}-J)_{nc}$, then $\frac{c_\alpha}{d_J}$ is even and hence that $\sum_{\alpha\in\widetilde{\pi}-J}\frac{c_\alpha}{2d_J}$ is an integer. We will use the notation and technique of Lemma 1.

First note that for each $\alpha\in\pi_c$ or π_{nc}, $\omega_\alpha\in\mathfrak{t}_{\mathbf{C}}$ and for each $\alpha\in\pi_{cx}$, $\theta\omega_\alpha=\omega_{\theta(\alpha)}$. Since, $\mathfrak{h}_{\mathbf{C}}^{\mathbf{L}_J}$ is the space of all sums of the form $\sum_{\alpha\in\widetilde{\pi}-J} d_\alpha\frac{2\omega_\alpha}{(\alpha,\ \alpha)}$, with each $d_\alpha\in\mathbf{C}$, such that $\sum_{\alpha\in\widetilde{\pi}-J} c_\alpha d_\alpha=0$, it is easy to check that $\mathfrak{t}_{\mathbf{C}}^{\mathbf{L}_J}$ is the set of all such sums satisfying the additional condition that for each $\alpha\in(\widetilde{\pi}-J)_{cx}$, $d_\alpha=d_{\theta(\alpha)}$.

Since $\frac{h_1}{2}=\frac{-2\pi i}{2d_J}\sum_{\alpha\in\widetilde{\pi}-J}\frac{2\omega_\alpha}{(\alpha,\ \alpha)}$, and $\sum_{\alpha\in\widetilde{\pi}-J}\frac{c_\alpha}{2d_J}$ is an integer, $\exp(\frac{h_1}{2})$ belongs to $Z_J\cap K_{\mathbf{C}}$.

Suppose that $\exp(\frac{h_1}{2})\notin(Z_J\cap K_{\mathbf{C}})_0$. Let M be a positive integer such that $2d_J$ and $\sum_{\alpha\in\widetilde{\pi}-J}\frac{c_\alpha}{2d_JM}$ are relatively prime. The element $\exp(\frac{h_1}{2M})\in Z_J\cap K_{\mathbf{C}}$, but $\notin(Z_J\cap K_{\mathbf{C}})_0$ (otherwise, $\exp(\frac{h_1}{2})\in(Z_J\cap K_{\mathbf{C}})_0$).

We have $\left(\exp(\frac{h_1}{2M})\right)^{d_J}=\exp(\frac{d_Jh_1}{2M})\in(Z_J)_0\cap K_{\mathbf{C}}$ because $Z_J/(Z_J)_0$ is cyclic of order d_J. We will show that $z=\exp(\frac{d_Jh_1}{2M})\notin(Z_J\cap K_{\mathbf{C}})_0$.

Suppose $z \in (Z_J \cap K_{\mathbf{C}})_0$. Then there exists $h = \sum_{\alpha \in \tilde{\pi} - J} d_\alpha \frac{2\omega_\alpha}{(\alpha,\ \alpha)} \in \mathfrak{t}_{\mathbf{C}}^{\mathbf{L}_J}$, such that $\beta(\frac{d_J h_1}{2M} - h) \in 2\pi i \mathbf{Z}$, for all $\beta \in \mathbf{Z}[\Delta_k]$. Arguing as in Lemma 1, we conclude that if, $\alpha \in (\tilde{\pi} - J)$ is imaginary, then for some integer s_α:

$$\frac{-2\pi i}{2M} - 2\pi i s_\alpha = d_\alpha. \tag{2.9}$$

Moreover, if $\alpha \in (\tilde{\pi} - J)_{cx}$, then

$$\frac{-2\pi i}{2M} - 2\pi i s_\alpha = d_\alpha. \tag{2.10}$$

But, if $\alpha \in (\tilde{\pi} - J)_{cx}$, then $d_\alpha = d_{\theta(\alpha)}$ and hence $s_\alpha = s_{\theta(\alpha)}$. Since $\sum_{\alpha \in \tilde{\pi} - J} c_\alpha d_\alpha = 0$, it follows that:

$$-2\pi i \frac{\sum_{\alpha \in \tilde{\pi} - J} c_\alpha}{2M} = 2\pi i \sum_{\alpha \in \tilde{\pi} - J} c_\alpha s_\alpha \tag{2.11}$$

and hence dividing both sides by $2\pi i$, we have

$$\frac{-\sum_{\alpha \in \tilde{\pi} - J} c_\alpha}{2M} = \sum_{\alpha \in \tilde{\pi} - J} c_\alpha s_\alpha \tag{2.12}$$

$$= \sum_{\alpha \in (\tilde{\pi} - J)_c} c_\alpha s_\alpha + \sum_{\alpha \in (\tilde{\pi} - J)_{nc}} c_\alpha s_\alpha + \sum_{\{\alpha,\ \theta(\alpha)\} \subset (\tilde{\pi} - J)_{cx}} 2 c_\alpha s_\alpha \tag{2.13}$$

Since $\frac{c_\alpha}{d_J}$ is even if $\alpha \in (\tilde{\pi} - J)_c$ or $(\tilde{\pi} - J)_{nc}$, and if $\alpha \in (\tilde{\pi} - J)_{cx}$, c_α is divisible by d_J, each sum on the right hand side is divisible by $2d_J$. It follows that $\frac{-\sum_{\alpha \in \tilde{\pi} - J} c_\alpha}{2M}$ is divisible by $2d_J$. This contradicts the fact that $2d_J$ and $\sum_{\alpha \in \tilde{\pi} - J} \frac{c_\alpha}{2d_J M}$ are relatively prime. This contradiction implies that $z \notin (Z_J \cap K_{\mathbf{C}})_0$. So if $\exp(\frac{h_1}{2}) \notin (Z_J \cap K_{\mathbf{C}})_0$, z is the desired element and we are done.

If $\exp(\frac{h_1}{2}) \in (Z_J \cap K_{\mathbf{C}})_0$ we can show that for some positive integer m,

$$\exp(\frac{h_1}{2(2d_J)^m}) \text{ belongs to } Z_J \cap K_{\mathbf{C}} \text{ but not } (Z_J \cap K_{\mathbf{C}})_0.$$

Then we can repeat the preceding argument with $\frac{h_1}{2(2d_J)^m}$ replacing $\frac{h_1}{2}$ to obtain a suitable element z.

Thus (i) $\Longrightarrow$ (iii).

We now show that (iii) $\Longrightarrow$ (i). In our proof that (iii) $\Longrightarrow$ (ii), we actually showed that (iii) implies that the 2 subgroup of $tor\left(\frac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ is equal to $\mathbf{Z}_2$. It follows from (2.5) that $tor\left(\frac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ has order $2d_J$. Thus $tor\left(\frac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ is either isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_{d_J}$ or $\mathbf{Z}_{2d_J}$. Since $\mathfrak{g}$ is not equal rank, $\mathfrak{g}$ is either classical or EI or EIV. So apriori using Sommers' calculations, we have $d_J = 1,\ 2$, or 3. If $d_J = 1$, then the torsion subgroup is $\mathbf{Z}_2$. $d_J = 3$ occurs only for $\mathfrak{g} = EI$ and $\mathfrak{g}_J$ is isomorphic to $sl(3,\ \mathbf{C}) \oplus su(2,\ 1)$. But then the 2 subgroup is trivial by (c).

So we need only consider $d_J = 2$. The torsion subgroup is either $\mathbf{Z}_2 \times \mathbf{Z}_2$ or $\mathbf{Z}_4$. The first alternative would contradict the hypothesis about the 2 subgroup. Hence there must be a cyclic generator in $tor\left(\frac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ of order 4. By the usual arguments, this generator has a representative of the form $\frac{n\psi_{\widetilde{\pi}-J}}{4} = \frac{n\tau_J}{2}$ for some integer n. Then n must be odd otherwise the generator has order less than 4. Therefore, this representative is equivalent to $\tau_J/2$, which must belong to $\mathbf{Z}[\Delta_k]$. We have shown that (iii) $\Longrightarrow$ (i). (The possibility we have just discussed only occurs for $\mathfrak{g} = EI$ and the semisimple part of $\mathfrak{g}_J$ isomorphic to $su(2,\ 2) \oplus sl(2,\ \mathbf{C})$.)

(c) See the proof of (b).

Clearly **(b)** $\Longrightarrow$ **(d)**.

This completes the proof of Proposition 4.

Corollary 1. *If $\mathfrak{g}$ is simple, then $Z_J \cap K_{\mathrm{C}}/(Z_J \cap K_{\mathrm{C}})_0$ is cyclic.*

Proof. If $\mathfrak{g}$ is equal rank, then $Z_J \subset T_{\mathrm{C}} \subset K_{\mathrm{C}}$. Hence $Z_J \cap K_{\mathrm{C}}/(Z_J \cap K_{\mathrm{C}})_0$ is equal to $Z_J/(Z_J)_0$, which Sommers has shown to be cyclic. If $\mathfrak{g}$ is not equal rank the desired conclusion follows from (2.5), Proposition 3, Proposition 4 and the fact that the quotient $Z_J/(Z_J)_0$ is cyclic.

Definition 6. *A subalgebra $\mathfrak{l} \subset (\mathfrak{g}_{\mathrm{C}})_J$ (resp., subgroup $L \subset G_{\mathrm{C}}$) is an* **elliptic pseudo Levi subalgebra** *(resp.,* **elliptic pseudo Levi subgroup***) of $\mathfrak{g}_{\mathrm{C}}$ (resp., G_{C}) if $\mathfrak{l}$ (resp., L) is conjugate under K_{C} to a standard elliptic pseudo Levi subalgebra (resp., standard elliptic pseudo Levi subgroup) of $\mathfrak{g}_{\mathrm{C}}$ (resp., G_{C}).*

Proposition 5. *(1) $\mathfrak{l} \subset \mathfrak{g}_{\mathrm{C}}$ is an elliptic pseudo Levi subalgebra of $\mathfrak{g}_{\mathrm{C}}$ if and only if there exists an element $z \in K$ such that $\mathfrak{l} = \mathfrak{g}_{\mathrm{C}}^z$.*

(2) $L \subset G_{\mathrm{C}}$ is an elliptic pseudo Levi subgroup of G_{C} if and only if there exists an element $z \in K$ such that L is the connected component of the identity of G_{C}^z, i.e. $L = (G_{\mathrm{C}}^z)_0$.

Proof. (1) and (2) of the proposition are equivalent so we only prove (1).

$\Longrightarrow$. It suffices to show that for each $J \subsetneq \widetilde{\pi}$, there exists $z \in K$ such that $(\mathfrak{g}_{\mathrm{C}})_J = \mathfrak{g}_{\mathrm{C}}^z$. We follow the proof of Proposition 2 in [14] with a few modifications.

We have

$$Z_J \cap K_{\mathrm{C}} = T_{\mathrm{C}}{}^{\mathbf{L}_J} = T_{\mathrm{C}}{}^{\mathbf{L}_J(\mathfrak{t})} = \exp(\mathfrak{t}_{\mathbf{C}}^{\mathbf{L}_J}) \times F,$$

where F is a finite cyclic subgroup of $Z_J \cap K_{\mathrm{C}}$ of order d_J^K. The generator of F has the form $\exp(v + iw)$ where $v, w \in \mathfrak{t}$. Since $\exp(v + iw) \in Z_J \cap K_{\mathrm{C}}$, if $\phi \in J$, then $e^{\phi}(\exp(v + iw)) = e^{\phi(v+iw)} = 1$. But $\phi(iw) \in \mathbf{R}$. It follows that $\phi(iw) = 0$ for all $\phi \in J$. Thus, without loss of generality, we may assume that F is generated by $\exp(v)$.

As Sommers notes, a maximal compact torus $\widehat{S}$ contained in an algebraic torus S contains an element y such that y^k $(k \in \mathbf{Z},\ k \neq 0)$ is not in the kernel of any non-trivial character of S. Such an element is called a toplogical generator for $\widehat{S}$. So let x_1 be a topological generator of $\exp(\mathfrak{t}_{\mathbf{C}}^{\mathbf{L}_J}) \cap K$, and set $x = x_1 \exp(v)$. By Lemma 1 in [14], $\mathbf{L}_J(\mathfrak{t})$ corresponds to the set of all of the characters of T_{C} which are trivial on x. It is clear that each root in Δ_J is trivial on x. We now show that Δ_J contains all roots in Δ which are trivial on x. This will follow from:

Lemma 5. *$\phi \in \Delta$ and $\phi^{\sharp} \in \mathbf{L}_J(\mathfrak{t})$, then $\phi \in \Delta_J$.*

Lemma 5 is proven like Lemma 4.

$\Longleftarrow$. Assume that $\mathfrak{l} = \mathfrak{g}^z_{\mathbf{C}}$ with $z \in K$. We must show that $\mathfrak{l}$ is conjugate under $K_{\mathbf{C}}$ to a standard elliptic pseudo Levi subalgebra. We will adapt an argument of Sommers based on the affine Weyl group, $\mathbf{W}_a$. We begin by recalling some facts about $\mathbf{W}_a$. If $\phi \in \Delta$, the coroot $H_\phi = \check{\phi} \in \mathfrak{h}_{\mathbf{C}}$ is defined to be $\dfrac{2h_\phi}{(\phi,\ \phi)}$. $\check{\Delta}$ denotes the coroots of Δ. Set $\check{\mathbf{L}}$ equal to the $\mathbf{Z}$ lattice spanned by $\check{\Delta}$. Then, $\mathbf{W}_a$ is the semidirect product $\mathbf{W} \ltimes \check{\mathbf{L}}$. For each, $\phi \in \Delta$, $k \in \mathbf{Z}$, define the affine reflection

$$s_{\phi,\ k}(x) = s_\phi(x) + kH_\phi.$$

We may also think of each pair $(\phi,\ n)$ with $\phi \in \Delta$ and $n \in \mathbf{Z}$ as defining an affine root. In this case, if $y \in \mathfrak{h}_{\mathbf{C}}$ then by definition $(\phi,\ n)(y) = \phi(y) - n$.

Conjugating by K, we can assume that $z \in T$. Let $\Phi = \{\beta \in \Delta |\ e^\beta(x) = 1\}$. Then, $\mathfrak{l} = \mathfrak{h} \bigoplus\limits_{\beta \in \Phi} (\mathfrak{g}_{\mathbf{C}})_\beta$.

We have $z = \exp(-2\pi i v)$ for some $v \in i\mathfrak{t}$. Pick a θ-stable simple system π for $\Delta(\mathfrak{g},\ \mathfrak{h})$. Let $\mathfrak{D} = \{h \in i\mathfrak{t} + \widetilde{a} |\ \psi(h) < 1,\ \text{and}\ \alpha(h) > 0,\ \text{for all}\ \alpha \in \pi\}$. The closure of $\mathfrak{D}$ is a fundamental domain for the action of $\mathbf{W}_a$ on $i\mathfrak{t} + \widetilde{a}$.

Let $\mathbf{W}'_a$ denote the subgroup of $\mathbf{W}_a$ generated by elements of the form $s_{\alpha,\ m}$, where α is imaginary and $m \in \mathbf{Z}$, and products $s_{\beta,k}s_{\theta\beta,k}$ where β is complex, the inner product of β and $\theta\beta$ is zero, and $k \in \mathbf{Z}$). Note that $\mathbf{W}'_a$ preserves $i\mathfrak{t}$. $\mathbf{W}^\theta \subset \mathbf{W}'_a$ and $\mathbf{W}^\theta$ is generated by elements of the form s_α, where α is imaginary, and products $s_\beta s_{\theta\beta}$ where β is complex and the inner product of β and $\theta\beta$ is zero.

To complete the proof of the implication $\Longleftarrow$, we need the following result.

Lemma 6. *There is an element in $\mathbf{W}'_a$ which conjugates v into the closure of $\mathfrak{D}$.*

Proof of Lemma. To simplify the details of the proof let us assume the $\phi(v) \notin \mathbf{Z}$ for all $\phi \in \Delta$. To prove Lemma 6, we must study the proof of Lemma 7.157 and modify part of the proof of Lemma 7.158 in [8].

Choose any $v_0 \in \mathfrak{D}$ and then find $w^* \in \mathbf{W}'_a$ so that the length $\|wv - v_0\|^2$ is minimal for $w \in \mathbf{W}'_a$. We claim that $w^*v \in \mathfrak{D}$. Otherwise, the proof of Lemma 7.157 shows that either (a) there is a simple root α such that $\alpha(v) < 0$ or (b) $\dfrac{-2\psi(v)}{(\psi,\ \psi)} + 1 < 0$. Suppose that (a) is true. If α is imaginary, then the proof of Lemma 7.158 shows that $\|s_{\alpha,0}w^*v - v_0\|^2 < \|w^*v - v_0\|^2$. If α is complex, then either (i) $(\alpha,\ \theta\alpha) < 0$, in which case $\gamma = \alpha + \theta\alpha$ is an imaginary root, $\gamma(v) < 0$ and $\|s_{\gamma,0}w^*v - v_0\|^2 < \|w^*v - v_0\|^2$ or (ii) $(\alpha,\ \theta\alpha) = 0$ in which case

$$\|s_{\alpha,0}s_{\theta\alpha,0}w^*v - v_0\|^2 < \|w^*v - v_0\|^2.$$

So in case (a) we obtain a contradiction. Similarly, alternative (b) leads to a contradiction. It follows that $w^*v \in \mathfrak{D}$. This completes the proof of Lemma 6.

We now complete the proof of Proposition 5 by imitating the end of Sommers' argument in Proposition 2 in [14]. It follows from Lemma 6 that up to conjugacy by $\mathbf{W}'_a$, we can assume that $v \in \overline{\mathfrak{D}}$. Let J be the roots of $\widetilde{\pi}$ which are integral on v. Then J is θ-stable and Δ_J are the roots of Δ which are integral on v. Hence Δ_J is $\mathbf{W}^\theta$-conjugate to Φ. Thus, for some $w \in \mathbf{W}^\theta$, $\mathfrak{l}$ is conjugate under $K_{\mathbf{C}}$ to $\mathfrak{h} \bigoplus\limits_{\beta \in w(\Delta_J)} (\mathfrak{g}_{\mathbf{C}})_\beta = (\mathfrak{g}_{\mathbf{C}})_{wJ}$.

Remark 4. *An elliptic pseudo Levi subalgebra* $\mathfrak{l}$ *must contain the complexification of a fundamental Cartan subalgebra of* $\mathfrak{g}$. *Thus, if* $\mathfrak{g}$ *is equal rank, then* $\mathfrak{l}$ *is equal rank. Hence in the equal rank case,* $\mathfrak{l}$ *must have a Cartan subalgebra which is a Cartan subalgebra of* $\mathfrak{k}_{\mathbf{C}}$. *It follows that in this case* $z(\mathfrak{l}) \subset \mathfrak{k}_{\mathbf{C}}$ *and that* $Z(L)\ /\ Z(L)_0 = Z(L) \cap K_{\mathbf{C}}/(Z(L) \cap K_{\mathbf{C}})_0$.

Corollary 2. *If* $\mathfrak{l}$ *is an elliptic pseudo Levi subalgebra of a simple Lie algebra* $\mathfrak{g}$, *and* L *is the corresponding connected subgroup of* $G_{\mathbf{C}}$, *then* $\dfrac{Z(L) \cap K_{\mathbf{C}}}{(Z(L) \cap K_{\mathbf{C}})_0}$ *is cyclic.*

Proof. This follows from Corollary 1 and Proposition 5.

Corollary 3. *If* $\mathfrak{l}$ *is an elliptic Levi subalgebra of* $\mathfrak{g}_{\mathbf{C}}$ *then* $Z(L)$ *is connected and* $Z(L) \cap K_{\mathbf{C}}$ *is connected.*

Proof. $Z(L)$ is connected by the discussion in section 1.2 of [14].

If $\mathfrak{g}$ is simple and equal rank then $Z(L) \cap K_{\mathbf{C}} = Z(L)$ so there is nothing to prove. If $\mathfrak{g}$ is simple and not equal rank, then $Z(L) \cap K_{\mathbf{C}}$ is connected by Proposition 5(a).

Definition 7. *If* $(\mathfrak{g}_{\mathbf{C}})_J$ *is a standard elliptic pseudo Levi subalgebra, let* d_J^K *denote the cardinality of the cyclic group* $\dfrac{Z_J \cap K_{\mathbf{C}}}{(Z_J \cap K_{\mathbf{C}})_0}$.

The following result will be needed in the proof of Proposition 7 below.

Proposition 6. *Here is a list of all the simple real algebras* $\mathfrak{g}$ *for which* $\mathfrak{g}_{\mathbf{C}}$ *contains an elliptic pseudo Levi subalgebra* $(\mathfrak{g}_{\mathbf{C}})_J$ *such that (1)* $d_J^K > 2$ *and (2)* $(\mathfrak{g}_{\mathbf{C}})_J \cap \mathfrak{p}_{\mathbf{C}}$ *has a noticed element. After each real form* $\mathfrak{g}$, *we list the* $G_{\mathbf{C}}$ *conjugacy classes of the* $\mathfrak{g}_J$. *Each* $\mathfrak{g}_J$ *is described by giving the isomorphism type of its semisimple part,* $\mathfrak{g}_J^{ss}$. *The value of* d_J^K *appears in parenthesis after* $\mathfrak{g}_J^{ss}$. *Note that all the real forms in the following list are quasisplit.*

EI: $su(2,1) \oplus sl(3, \mathbf{C})$ *(3);* $su(2,2) \oplus sl(2, \mathbf{C})$ *(4).*

EII: $3su(2,1)$ *(3).*

EV: $su(3,3) \oplus su(2,1)$ *(3);* $2su(2,2) \oplus sl(2, \mathbf{R})$ *(4);* $3su(2,1)$ *(3).*

EVI: $3su(2,1)$ *(3).*

EVIII: $2su(3,2)$ *(5);* $su(3,3) \oplus su(2,1) \oplus sl(2, \mathbf{R})$ *(6);* $su(4,4) \oplus sl(2, \mathbf{R})$ *(4);* $so(6,4) \oplus su(2,2)$ *(4);* $su(5,4)$ *(3); EII* $\oplus\ su(2,1)$ *(3);* $su(3,3) \oplus su(2,1)$ *(3);* $2su(2,2) \oplus sl(2, \mathbf{R})$ *(4);* $3su(2,1) \oplus sl(2, \mathbf{R})$ *(3);* $3su(2,1)$ *(3).*

EIX: $3su(2,1)$ *(3).*

FI: $su(2,1) \oplus \widetilde{su(2,1)}$ *(3);* $su(2,2) \oplus \widetilde{sl(2, \mathbf{R})}$ *(4).*

GI (split real form of G2): $su(2,1)$ *(3).*

Note: $\widetilde{su(2,1)}$ *and* $\widetilde{sl(2, \mathbf{R})}$ *indicate that the corresponding roots in* J *are short.*

Proof. Case by case verification. The verification is based on two essential facts. First, Sommers [13] has determined the $G_{\mathbf{C}}$ conjugacy classes of pseudo Levi subalgebras of $\mathfrak{g}_{\mathbf{C}}$ for which $d_J > 2$. Second, from [10] one knows the simple $\mathfrak{g}$ for which $\mathfrak{g}_{\mathbf{C}} \cap \mathfrak{p}_{\mathbf{C}}$ contains noticed elements.

Assume $\mathfrak{g}$ is classical. If $\mathfrak{g}$ is equal rank, then for each J, $d_J = d_J^K$. Sommers has shown that for any pseudo Levi subalgebra $(\mathfrak{g}_{\mathbf{C}})_J$, $d_J \leq 2$. If $\mathfrak{g}$ is not equal rank then by applying Proposition 4, we find that in all cases $d_J^K \leq 2$.

Assume $\mathfrak{g}$ is exceptional. If $\mathfrak{g}$ is equal rank, then for each J, $d_J = d_J^K$. Sommers has determined all the ($G_{\mathbf{C}}$ conjugacy classes of) $(\mathfrak{g}_{\mathbf{C}})_J$ for which $d_J \geq 3$. Using

[10] one can determine which equal rank real forms of $(\mathfrak{g}_{\mathbf{C}})_J$ are contained in $\mathfrak{g}$ such that $(\mathfrak{g}_{\mathbf{C}})_J$ contains noticed elements. If $\mathfrak{g}$ is not equal rank then $\mathfrak{g} = EI$, or EIV. We now apply Proposition 4. If $\mathfrak{g} = EIV$, then $d_J^K \leq 2$ in all cases. If $\mathfrak{g} = EI$, then we find the semisimple part of $\mathfrak{g}_J$ is either $su(2,\ 1) \oplus sl(3,\ \mathbf{C})$ or $su(2,\ 2) \oplus sl(2,\ \mathbf{C})$.

Let $Out_\theta(\mathfrak{g}_{\mathbf{C}})$ denote the group of outer automorphisms of $\mathfrak{g}_{\mathbf{C}}$ which commute with θ. The following lemma concerns the effect of automorphisms in $Out_\theta(\mathfrak{g}_{\mathbf{C}})$ on the noticed $K_{\mathbf{C}}$ nilpotent conjugacy classes of certain simple algebras. The lemma will be used in the proof of Proposition 7.

Lemma 7. *(a) If e is a noticed nilpotent in $\mathfrak{p}_{\mathbf{C}}$, and $T \in Out_\theta(\mathfrak{g}_{\mathbf{C}})$, then $T(e)$ is a noticed nilpotent.*

(b)If $\mathfrak{g} = EI$, EII, $sl(3,\ \mathbf{C})$, or $su(p,\ q)$, then any automorphism in $Out_\theta(\mathfrak{g}_{\mathbf{C}})$ fixes each $K_{\mathbf{C}}$-conjugacy class of noticed nilpotents in $\mathfrak{p}_{\mathbf{C}}$.

Proof. (a) $T(K_{\mathbf{C}} \cdot e) = K_{\mathbf{C}} \cdot e$. Let $\{x,\ e,\ f\}$ is a normal $sl(2)$ triple containing e. Since e is noticed, $\mathfrak{k}_{\mathbf{C}}^{\{x,\ e,\ f\}} = 0$. Since $\{T(x),\ T(e),\ T(f)\}$ is a normal triple containing $T(e)$ and $\mathfrak{k}_{\mathbf{C}}^{\{T(x),\ T(e),\ T(f)\}} = T^{-1}(\mathfrak{k}_{\mathbf{C}}^{\{x,\ e,\ f\}})$, we have $\mathfrak{k}_{\mathbf{C}}^{\{T(x),\ T(e),\ T(f)\}} = 0$.

(b) By (a), if $\mathcal{O}$ is a noticed class in $\mathfrak{p}_{\mathbf{C}}$, and $T \in Out_\theta(\mathfrak{g}_{\mathbf{C}})$, then $T(\mathcal{O})$ is also a noticed class. If $\mathfrak{g} = sl(3,\ \mathbf{C})$, there is only one $K_{\mathbf{C}}$ conjugacy class of noticed nilpotent in $\mathfrak{p}_{\mathbf{C}}$, the principal class. If $\mathfrak{g} = EI$, the only non trivial element of $Out_\theta(\mathfrak{g}_{\mathbf{C}})$ is θ itself, so there is nothing to prove. We will only consider the case of EII and organize the verification steps into the following example. We leave the case of $su(p,\ q)$ to the reader.

Example 5. $\mathfrak{g} = EII$. *Then from* [1] *and* [10]*, there are 4 noticed nilpotent orbits. Let use the notation of Appendix A. We can realize a system of simple roots for $\mathfrak{k}_{\mathbf{C}} = \mathfrak{sl}_6(\mathbf{C}) \oplus \mathfrak{sl}_2(\mathbf{C})$ using $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ for the first summand and $\psi = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ for the second. Here the simple roots α_4 is noncompact imaginary. The non trivial automorphism T of $Out_\theta(\mathfrak{g}_{\mathbf{C}})$ fixes α_2 and α_4 while interchanging α_1 with α_6 and α_3 with α_5. According to the table in* [1]*, the four noticed orbits have $K_{\mathbf{C}}$-labels 22222, 04040, 44044, 44444 on the sequence $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ respectively. Clearly such a labelling will not change under the action of T. Hence each noticed conjugacy class is preserved by T.*

Definition 8. *Suppose e is a nilpotent in $\mathfrak{p}_{\mathbf{C}}$. Let $\mathcal{L}(e)$ denote the set of all triples $(\mathfrak{l},\ \mathfrak{q},\ \overline{M_{K_{\mathbf{C}}} \cdot e})$, where $\mathfrak{l}$ is an elliptic pseudo Levi subalgebra of $\mathfrak{g}_{\mathbf{C}}$ in which e is noticed and $\mathfrak{q} \subset \mathfrak{l}$ is a θ-stable noticed parabolic for e. Here $\mathfrak{q}$ has Levi decomposition $\mathfrak{m} \oplus \mathfrak{u}$; $M_{K_{\mathbf{C}}}$ is the connected subgroup of L with Lie algebra $\mathfrak{m} \cap \mathfrak{k}_{\mathbf{C}}$; and " $\overline{}$ " denotes closure.*

The next major result we need is the following analog of Proposition 9 in Sommers thesis.

Proposition 7. *Let J be as usual. Assume that $((\mathfrak{g}_{\mathbf{C}})_J,\ \mathfrak{q},\ \overline{M_{K_{\mathbf{C}}} \cdot e}) \in \mathcal{L}(e)$ Then the triple $((\mathfrak{g}_{\mathbf{C}})_J,\ \mathfrak{q},\ \overline{M_{K_{\mathbf{C}}} \cdot e})$ determines a well defined conjugacy class in $A_k(e)$.*

The orbit $(K_{\mathbf{C}})_J \cdot e$ in $(\mathfrak{g}_{\mathbf{C}})_J \cap \mathfrak{p}_{\mathbf{C}}$ corresponds to the triple $((\mathfrak{g}_{\mathbf{C}})_J,\ \mathfrak{q},\ \overline{M_{K_{\mathbf{C}}} \cdot e})$ in $\mathcal{L}(e)$.

Under the hypotheses of Proposition 7, it is clear that if the coset $z(Z_J \cap K_{\mathbf{C}})_0$ generates $Z_J \cap K_{\mathbf{C}}/(Z_J \cap K_{\mathbf{C}})_0$, z defines an element in $A_k(e)$, namely the coset

$z(K^e_{\mathbf{C}})_0$. However, one must show that the conjugacy class of $z(K^e_{\mathbf{C}})_0$ in $A_k(e)$ depends only on the triple $((\mathfrak{g}_{\mathbf{C}})_J,\ \mathfrak{q},\ \overline{M_{K_{\mathbf{C}}}\cdot e})$.

The proof of Proposition 7 depends on several lemmas.

Lemma 8. *Assume (i) the hypotheses of Proposition 7 hold, (ii) the coset $z(Z_J \cap K_{\mathbf{C}})_0$ generates $Z_J \cap K_{\mathbf{C}}/(Z_J \cap K_{\mathbf{C}})_0$, and (iii) $d^K_J = 1$ or $d^K_J = 2$. Then the image of z determines a well defined conjugacy class in $A_k(e)$.*

Proof. By hypothesis $Z_J \cap K_{\mathbf{C}}/(Z_J \cap K_{\mathbf{C}})_0$ is either (a) trivial or (b) $\mathbf{Z}_2$. Suppose $z_1(Z_J \cap K_{\mathbf{C}})_0$ and $z_2(Z_J \cap K_{\mathbf{C}})_0$ each generate $Z_J \cap K_{\mathbf{C}}/(Z_J \cap K_{\mathbf{C}})_0$. In case (a), z_1 and z_2 both determine the trivial conjugacy class. In case (b), $z_1 = z_2 a$ where $a \in (Z_J \cap K_{\mathbf{C}})_0 \subset (K^e_{\mathbf{C}})_0$, so that z_1 and z_2 determine the same element in $A_k(e)$.

In light of Lemma 8, to complete the proof of Proposition 7, it suffices to consider the triples in $\mathcal{L}(e)$ containing $(\mathfrak{g}_{\mathbf{C}})_J$ with $d^K_J \geq 3$. We divide the set of such $(\mathfrak{g}_{\mathbf{C}})_J$ into three groups.

Group A: The following simple $\mathfrak{g}$ and $G_{\mathbf{C}}$ classes of elliptic pseudo Levi subalgebras. Each $(\mathfrak{g}_{\mathbf{C}})_J$ is described by giving $\mathfrak{g}^{ss}_J$. The cardinality of d^K_J appears in parenthesis after $\mathfrak{g}^{ss}_J$.

EI: $su(2,\ 1) \oplus sl(3,\ \mathbf{C})$ (3).

EII: $3su(2,\ 1)$ (3).

EVI: $3su(2,\ 1)$ (3).

EVIII: $su(3,\ 3) \oplus su(2,\ 1) \oplus sl(2,\ \mathbf{R})$ (6); $su(5,\ 4)$ (3); $EII \oplus su(2,\ 1)$ (3); $su(3,\ 3) \oplus su(2,\ 1)$ (3); $3su(2,\ 1)$ (3).

EIX: $3su(2,\ 1)$ (3).

FI: $su(2,\ 1) \oplus \widetilde{su(2,\ 1)}$ (3); $su(2,\ 2) \oplus \widetilde{sl(2,\ \mathbf{R})}$ (4).

GI: $su(2,\ 1)$ (3).

Group B: The following simple $\mathfrak{g}$ and $G_{\mathbf{C}}$ classes of elliptic pseudo Levi subalgebras. The relevant nilpotent conjugacy classes of $\mathfrak{p}_{\mathbf{C}}$ which are noticed in $(\mathfrak{g}_{\mathbf{C}})_J$ are listed in parenthesis after $(\mathfrak{g}_{\mathbf{C}})_J$. The information about $A_k(e)$ comes from [6].

EI: $su(2,\ 2) \oplus sl(2,\ \mathbf{C})$ (4) [class 23 - noticed, $A_k(e) = S_4$].

EV: $3su(2,\ 1)$ (3) (class 27); $su(3,\ 3) \oplus su(2,\ 1)$ (3) (classes 69 and 70)

EVIII: $2su(3,\ 2)$ (5) (class 68); $3su(2,\ 1) \oplus sl(2,\ \mathbf{R})$ (3) (class 25); $so(6,\ 4) \oplus su(2,\ 2)$ (4) [class 67 ($A_k(e) = D(4)$), class 68 ($A_k(e) = S_5$)–distinguished in EVIII].

Group C: The following simple $\mathfrak{g}$ and $G_{\mathbf{C}}$ classes of elliptic pseudo Levi subalgebras. The relevant nilpotent conjugacy classes of $\mathfrak{p}_{\mathbf{C}}$ which are noticed in $(\mathfrak{g}_{\mathbf{C}})_J$ are listed in parenthesis after $(\mathfrak{g}_{\mathbf{C}})_J$. In each case $A_k(e) = \mathbf{Z}_4$. This information comes from [6].

EV: $2su(2,\ 2) \oplus sl(2,\ \mathbf{R})$ (4) (class 50 - noticed in *EV*).

EVIII: $su(4,\ 4) \oplus sl(2,\ \mathbf{R})$ (4) (class 88-noticed in *EVIII*); $so(6,\ 4) \oplus su(2,\ 2)$ (4) (class 85 - distinguished in *EVIII*); $2su(2,\ 2) \oplus sl(2,\ \mathbf{R})$ (4) (class 38).

We introduce several notions that will be useful in the verification of Proposition 7.

Definition 9. *Assume that $\mathfrak{g}$ is equal rank, quasisplit and that Δ^+_k is a fixed positive root system for $(\mathfrak{k}_{\mathbf{C}},\ \mathfrak{t}_{\mathbf{C}})$. Suppose that Δ^+ is compatible with Δ^+_k, and π is the set of simple roots for Δ^+. Then, π is said to be special if each root in π is noncompact. Δ^+ and the Borel subalgebra of $\mathfrak{g}_{\mathbf{C}}$ which it determines are also said to be special.*

We will often make use of the following observation. It is a consequence of Corollary 5.8 and Theorem 6.2 in [3].

Remark 5. *Special simple root systems exist for $\mathfrak{g}_{\mathbb{C}}$ if and only if $\mathfrak{g}$ is equal rank and quasisplit. For EII and $EVIII$ there is a unique special simple root system denoted π_I. For EV, there are two such special simple root systems denoted by π_I and π_{II}. These special simple root systems are described in Appendix A.*

Verification of Proposition 7 for $(\mathfrak{g}_{\mathbb{C}})_J$ in Group A.

Lemma 9. *If $(\mathfrak{g}_{\mathbb{C}})_J$ belongs to Group A above then there exists $w \in \mathbf{W}_k$ such that $w(J) = J$ and*

(1) If $\mathfrak{g}$ is equal rank, the action of w on $\mathbf{L}/\mathbf{L}_J$ generates the automorphism group of the torsion subgroup of $\mathbf{L}/\mathbf{L}_J$.

(2) If $\mathfrak{g} = EI$, the action of w on $\dfrac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}$ generates the automorphism group of $tor\left(\dfrac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$.

Definition 10. *(Sommers element) Assume that $\mathfrak{g}$ is simple and Δ_k^+ is a fixed positive root system for $(\mathfrak{k}_{\mathbb{C}},\ \mathfrak{t}_{\mathbb{C}})$, and Δ^+ is a θ-stable compatible system for Δ. As usual π is the set of simple roots for Δ^+, $J \subsetneq \widetilde{\pi}$. w_0 denotes the longest element of $\mathbf{W}$. $(w_0)_K$ denotes the longest element of $\mathbf{W}_k$. $(w_0)_J$ denotes the longest element of $\mathbf{W}_J$. The Sommers' element $w_S = w_S(\pi,\ J)$ associated to J is defined to be $w_0(w_0)_J$.*

Remark 6. *If $\mathfrak{g} = EVI,\ EVIII,\ EIX,\ FI$, or GI, then for any choice of π compatible with Δ_k^+, $w_0 = (w_0)_K = -1$. If $\mathfrak{g} = EII$, then there exists π such that $w_0 = (w_0)_K \neq -1$. (See subcase (2) of proof of Lemma 9.)*

Proof of Lemma 9. Case by case verification. We show that for each $\mathfrak{g}_J^{ss}$ at issue, there exists a θ-stable system π and a subset $J \subsetneq \widetilde{\pi}$ such that the Sommers element $w_S = w_S(J,\ \pi) \in \mathbf{W}_k$. By the proof of Proposition 8 in [14], the element $w = w_S$ satisfies the assertion in the lemma.

We divide the verification of the existence of a suitable Sommers element into three parts. Subcase (1): $\mathfrak{g} \neq EII$, and each simple component of $\mathfrak{g}_J^{ss}$ is of type $su(2,\ 1)$. Subcase (2): $\mathfrak{g} = EII$ and $\mathfrak{g}_J^{ss}$ is of type $3su(2,\ 1)$. Subcase (3): at least one simple component of $\mathfrak{g}_J^{ss}$ is not of type $su(2,\ 1)$.

Subcase (1) occurs for $\mathfrak{g} = EVI,\ EVIII,\ EIX,\ FI$ and GI. Suppose $J \subsetneq \widetilde{\pi}$ for some θ-stable simple system π. Since $J = J_1 \cup J_2 \cup \ldots J_s$ (disjoint union) where each J_i corresponds to an $su(2,\ 1)$ factor, there exists w in the Weyl group of $((\mathfrak{g}_J^{ss})_{\mathbf{C}},\ (\mathfrak{g}_J^{ss})_{\mathbf{C}} \cap \mathfrak{t}_{\mathbb{C}})$ such that the simple roots in wJ are all noncompact. Since $w\pi$ is also a θ-stable system, we might as well assume that all the roots in J are noncompact. Therefore, for each i, $J_i = \{\phi_1^i,\ \phi_2^i\}$ where ϕ_1^i and ϕ_2^i are each noncompact. Since $\phi_1^i + \phi_2^i$ is compact, the longest element of the Weyl group of each J_i belongs $\mathbf{W}_k$. Therefore, $(w_0)_J \in \mathbf{W}_k$. Since $w_0 \in \mathbf{W}_k$ by Remark 6, $w_S = w_S(\pi,\ J) \in \mathbf{W}_k$.

Subcase (2): $\mathfrak{g} = EII$ and $\mathfrak{g}_J$ is of type $3su(2,\ 1)$. Let π_0 and π_I be the simple systems for $\mathfrak{g}$ given in Appendix A. Let ψ_0 (resp., ψ_I) be the longest root of the positive system determined by π_0 (resp., π_I). ψ_0 is compact and ψ_I is noncompact. $\{\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \ \psi_0\}$ comprises a simple system for $\mathfrak{k}_{\mathbb{C}}$. There is a single $K_{\mathbb{C}}$

conjugacy class of elliptic pseudo Levi subalgebras of type $3su(2,\ 1)$. (See Appendix B.) We can construct a representative of this conjugacy class by choosing $J \subset \widetilde{\pi_I}$, i.e., $J = \{\beta_1,\ \beta_3\} \cup \{\beta_2,\ -\psi_I\} \cup \{\beta_5,\ \beta_6\}$. As in subcase (1), we have $(w_0)_J \in \mathbf{W}_k$ since ψ_I as well as all the β_i are noncompact. One checks that w_0, the longest element of $\mathbf{W}$ relative to the simple system π_I, is equal to the longest element of $\mathbf{W}_k$ for the simple system given for $\mathfrak{k}_{\mathbf{C}}$. It follows that the Sommers element $w_S = w_0(w_0)_J \in \mathbf{W}_k$.

Subcase (3). There is a single $K_{\mathbf{C}}$ conjugacy class of elliptic pseudo Levi subalgebras $\mathfrak{l}$ such that $\mathfrak{l}_{\mathbf{R}}$ is of the given isomorphism type. One checks that a representative $\mathfrak{g}_J$ of this conjugacy class can be chosen so that $J \subset \widetilde{\pi_I}$ where π_I is defined as in Appendix A. One also checks that the corresponding Sommers element w_S lies in $\mathbf{W}_k$. In most cases checking that w_S lies in $\mathbf{W}_k$ requires computer computation. We will look at a few examples which do not require such checking.

Example 6. *Suppose $\mathfrak{g}_J$ is a representative of the elliptic pseudo Levi subalgebra of type $EII + su(2,\ 1)$ in $EVIII$. Then we can assume that $J \subsetneq \widetilde{\pi_I}$ where π_I is the unique special simple system for $EVIII$. $J = J_1 \cup J_2$ where J_1 is of type EII and J_2 is of type $su(2,\ 1)$. J_1 is a special simple system of EII, and J_2 is a special simple system of $su(2,\ 1)$. $(w_0)_J = (w_0)_{J_1}(w_0)_{J_2}$. It follows from the discussion of EII in subcase (2), that $(w_0)_{J_1} \in \mathbf{W}_k$. The discussion in subcase (1) implies that $(w_0)_{J_2} \in \mathbf{W}_k$. Since $(w_0) = -1 \in \mathbf{W}_k$, we see that $w_S(J,\ \pi_I) \in \mathbf{W}_k$.*

Lastly, we consider $\mathfrak{g} = EI$ and $\mathfrak{g}_J$ of type $su(2,\ 1) \oplus sl(3,\ \mathbf{C})$. EI is not equal rank so the previous arguments must be modified. We may assume that $J \subsetneq \widetilde{\pi_I}$. It happens that in this case the restriction of the Sommers element $w_S(J,\ \pi_I)$ to $\mathfrak{t}_{\mathbf{C}}$ is an element w of $\mathbf{W}_k$. Since $tor\left(\dfrac{\mathbf{Z}[\Delta_k]}{\mathbf{L}_J(\mathfrak{t})}\right)$ is isomorphic to the torsion subgroup of $\mathbf{L}/\mathbf{L}_J$, this suffices to establish (2) of Lemma 9.

To complete the verification of Proposition 7 for $(\mathfrak{g}_{\mathbf{C}})_J$ in Group A, we can imitate Sommers' proof of Proposition 9 in [14].

Note that the images of z_1 and z_2 both generate $Z_J \cap K_{\mathbf{C}}/(Z_J \cap K_{\mathbf{C}})_0$. Then $z_2 \equiv z_1^l$ modulo $(Z_J \cap K_{\mathbf{C}})_0$ for some l prime to d_J^K. By Lemma 9, we can find $w^{-1} \in \mathbf{W}_k$ such that $w^{-1}(J) = J$, and w^{-1} acts on the torsion subgroup of $\mathbf{L}/\mathbf{L}_J$ by multiplying each element by l. If $\bar{w}$ in $K_{\mathbf{C}}$ represents w, we have $Ad(\bar{w})z_1 \equiv z_1^l \equiv z_2$ modulo $(Z_J \cap K_{\mathbf{C}})_0$.

In addition, $Ad(\bar{w})$ is a θ stable outer automorphism of $(\mathfrak{g}_{\mathbf{C}})_J$. Using Lemma 7, one checks case by case that the θ stable outer automorphisms of $(\mathfrak{g}_{\mathbf{C}})_J$ preserve each $(K_{\mathbf{C}})_J$ conjugacy class of noticed elements. Since $Ad(\bar{w})e$ and e belong to the same $(K_{\mathbf{C}})_J$ conjugacy class, there exists $k \in (K_{\mathbf{C}})_J$ such that $Ad(k\bar{w})e = e$, i.e, $k\bar{w} \in K_{\mathbf{C}}^e$. Since $z_2 \in Z_J \cap K_{\mathbf{C}}$, and $k \in (K_{\mathbf{C}})_J$, we have $Ad(k\bar{w})z_1 \equiv z_2$ modulo $(Z_J \cap K_{\mathbf{C}})_0$. Since $(Z_J \cap K_{\mathbf{C}})_0 \subset ({K_{\mathbf{C}}}^e)_0$, we have shown that z_1 and z_2 determine the same conjugacy class in $A_k(e)$.

This completes the verification of Proposition 7 for Group A.

Verification of Proposition 7 for $\mathfrak{g}_J$ in group B.

The argument of the following lemma can be applied in each of the cases in group B.

Lemma 10. *Proposition 7 holds if $\mathfrak{g} = EV$ with $\mathfrak{g}_J = su(3,\ 3) \oplus su(2,\ 1)$.*

Proof. $\mathfrak{g}_J = su(3,\ 3)\ \oplus\ su(2,\ 1)$. (There is one $K_{\mathbb{C}}$ conjugacy class of this type of subalgebra.) The nilpotent classes of EV which are noticed in $(\mathfrak{g}_{\mathbb{C}})_J$ are classes 69 and 70 in Djokovic's list. $Z_J \cap K_{\mathbb{C}}/(Z_J \cap K_{\mathbb{C}})_0 = \mathbf{Z}_3$. Classes 69 and 70 are distinguished (so that $\mathfrak{g} = EV$ is the minimal elliptic Levi subalgebra containing each one). From Djokovic's tables, each class is also noticed in an elliptic pseudo Levi subalgebra of type $so(6,\ 6) \oplus sl(2,\ \mathbf{R})$. (There is only one K conjugacy class of such algebras).

For both class 69 and 70, $A_k(e) = S_3$. (The reasoning is as follows. Since each class is distinguished, $G^{\{x,\ e,\ f\}}$ is discrete. From Sommers work, we know that $G^{\{x,\ e,\ f\}} = A(e) = S_3$. Since both classes lie in $su(3,\ 3)\ \oplus\ su(2,\ 1)$ and $so(6,\ 6)\ \oplus\ sl(2,\ \mathbf{R})$, $\mathbf{Z}_3$ and $\mathbf{Z}_2$ belong to $K^{\{x,\ e,\ f\}}$. It follows that $K^{\{x,\ e,\ f\}} = G^{\{x,\ e,\ f\}} = S_3$.) Thus, each generator of $Z_J \cap K_{\mathbb{C}}/(Z_J \cap K_{\mathbb{C}})_0 = \mathbf{Z}_3$ lies in the same conjugacy class of $A_k(e)$.

The other cases in group B are handled as in Lemma 10.

For, EI: $su(2,\ 2) \oplus sl(2,\ \mathbf{C})$ $(\mathbf{Z}_4)$ (class 23), we use the fact that $A_k(e) = S_4$ ([6]) which has only one conjugacy class of elements of order 4.

For $EVIII$: $3su(2,\ 1)\ \oplus\ sl(2,\ \mathbf{R})$ $(\mathbf{Z}_3)$ (class 25), we use the fact that, $A_k(e) = S_3$ ([6]).

$EVIII$: $so(6,\ 4) \oplus su(2,\ 2)$ (4) [class 67 ($A_k(e) = D(4)$)].

$EVIII$: $so(6,\ 4) \oplus su(2,\ 2)$ (4) [class 68 ($A_k(e) = S_5$)–distinguished].

$EVIII$: $2su(3,\ 2)$ (5) [class 68 ($A_k(e) = S_5$)–distinguished].

This completes the verification of Proposition 7 for Group B.

Verification of Proposition 7 for $\mathfrak{g}_J$ in Group C.

Consider first $\mathfrak{g}_J = 2su(2,\ 2) \oplus sl(2,\ \mathbf{R})$ in EV. There is one $K_{\mathbb{C}}$-conjugacy class of elliptic pseudo Levi subalgebras of type $2su(2,\ 2)\ \oplus\ sl(2,\ \mathbf{R})$. If e is noticed in the corresponding elliptic pseudo Levi subalgebra, then e belongs to $K_{\mathbb{C}}$ conjugacy class 50 in Djokovic's list. We have $Z_J = Z_J \cap K_{\mathbb{C}} = K_{\mathbb{C}}^{\{x,\ e,\ f\}} = A_k(e) = \mathbf{Z}_4$.

Suppose that $(\mathfrak{l}',\ \mathfrak{b}',\ W') \in \mathcal{L}(e)$, with $Z(L') = \mathbf{Z}_4$. There is an element of $k \in K_{\mathbb{C}}$ which conjugates this triple to $(\mathfrak{l},\ \mathfrak{b},\ W)$ where $\mathfrak{l}_{\mathbf{R}} = 2su(2,\ 2)\ \oplus\ sl(2,\ \mathbf{R})$, $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$ is a special Borel subalgebra of $\mathfrak{l}$ and $W = \mathfrak{n}/[\mathfrak{n},\ \mathfrak{n}]$. From Appendix A, π_I and π_{II} are the unique special simple systems of EV. From Appendix B, the simple roots of $\mathfrak{b}$ form a system of type $2A_3 + A_1$ and must be a subset of one of the following extended simple systems $\widetilde{\pi_I}$, $\widetilde{s_{\beta_2}\pi_I}$, $\widetilde{\pi_{II}}$ or $\widetilde{s_{\beta_2}\pi_{II}}$. Let these subsets of the extended simple systems just listed be denoted J_i', $i = 1, \ldots, 4$. It is known that no two of the simple systems π_I, $s_{\beta_2}\pi_I$, π_{II} and $s_{\beta_2}\pi_{II}$ are conjugate under $\mathbf{W}_k$. Let $w^\flat$ denote the automorphism of Δ defined by the assignment: $-\psi_I \mapsto -\beta_7$, $\beta_1 \mapsto \beta_5 + \beta_6 + \beta_7$, $\beta_3 \mapsto -\beta_5$, $-\beta_2 \mapsto \beta_2$, $\beta_5 \mapsto \beta_3$, $\beta_6 \mapsto \beta_1$ and $\beta_7 \mapsto -\psi_I$. Then one can check that $w^\flat \in \mathbf{W}_k$ (see Appendix B), $w^\flat(J_1') = J_3'$, and $w^\flat(J_2') = J_4'$. It follows that the triples $(\mathfrak{l},\ \mathfrak{b}_1,\ W_1)$ and $(\mathfrak{l},\ \mathfrak{b}_3,\ W_3)$ are conjugate under $K_{\mathbb{C}}$, and the triples $(\mathfrak{l},\ \mathfrak{b}_2,\ W_2)$ and $(\mathfrak{l},\ \mathfrak{b}_4,\ W_4)$ are conjugate under $K_{\mathbb{C}}$. At the same time $(\mathfrak{l},\ \mathfrak{b}_1,\ W_1)$ and $(\mathfrak{l},\ \mathfrak{b}_2,\ W_2)$ are not conjugate under $K_{\mathbb{C}}$ since π_I and $s_{\beta_2}\pi_I$ are not conjugate under $\mathbf{W}_k$. It follows that there are two $K_{\mathbb{C}}$ conjugacy classes in $\mathcal{L}(e)$. Thus one can match the pair of generators z and z^3 of $\mathbf{Z}_4$ to the two $K_{\mathbb{C}}$ conjugacy classes in $\mathcal{L}(e)$. (One can distinguish z and z^3 by the value taken by e^{β_4} on each element.)

For $EVIII$: $su(4,\ 4) \oplus sl(2,\ \mathbf{R})$ (class 88, noticed), we argue as above using the extended systems $\widetilde{\pi_I}$ and $\widetilde{s_{\beta_1}\pi_I}$. There are two $K_{\mathbb{C}}$ conjugacy classes in $\mathcal{L}(e)$. One can match the pair of generators z and z^3 of $\mathbf{Z}_4$ to the two $K_{\mathbf{C}}$ conjugacy classes in $\mathcal{L}(e)$.

For $EVIII$: $so(6,\ 4) \oplus su(2,\ 2)$ (class 85 - distinguished), we argue as above using the extended systems $\widetilde{\pi_I}$ and $\widetilde{s_{\beta_2}s_{\beta_5}\pi_I}$. We obtain the same conclusion as for the elliptic pseudo Levi subalgebra $su(4,\ 4) \oplus sl(2,\ \mathbf{R})$.

For $EVIII$: $2su(2,\ 2)\ \oplus\ sl(2,\ \mathbf{R})$ (class 38), we argue as above using the extended systems $\widetilde{\pi_I}$, $\widetilde{s_{\beta_1}\pi_I}$, $\widetilde{s_{\beta_2}s_{\beta_5}\pi_I}$, and $s_{\beta_2}\widetilde{s_{\beta_5}s_{\beta_1}}\pi_I$. It turns out that the Borel subalgebra of $2su(2,\ 2)\ \oplus\ sl(2,\ \mathbf{R})$ constructed on $\widetilde{\pi_I}$ is $\mathbf{W}_k$ conjugate to the Borel subalgebra of $2su(2,\ 2)\ \oplus\ sl(2,\ \mathbf{R})$ constructed on $s_{\beta_2}\widetilde{s_{\beta_5}s_{\beta_1}}\pi_I$. Likewise the Borel of $2su(2,\ 2) \oplus sl(2,\ \mathbf{R})$ constructed on $\widetilde{s_{\beta_1}\pi_I}$ is $\mathbf{W}_k$ conjugate to the one built on $\widetilde{s_{\beta_2}s_{\beta_5}\pi_I}$. One can choose the following element $w^{\#}$ of $\mathbf{W}_k$ to accomplish both conjugations. If we identify the real span of the roots of $EVIII$ with $\mathbf{R}^8$, and write the roots of π_0 (see Appendix A) as in Appendix C of [7], then $w^{\#}$ switches coordinates 1 and 2, 3 and 5, 4 and 6 and 7 and 8. Since $\mathbf{W}_k$ contains the permutations of the coordinates of a vector in $\mathbf{R}^8$, $w^{\#} \in \mathbf{W}_k$. On the other hand, it is easy to check that the Borel subalgebra constructed on $\widetilde{\pi_I}$ is not $\mathbf{W}_k$ conjugate to the one constructed on $\widetilde{s_{\beta_1}\pi_I}$. We conclude as in the previous cases in group C that there are exactly two $K_{\mathbf{C}}$ conjugacy classes in $\mathcal{L}(e)$ which are matched to the two generators of $A_k(e)$.

This completes the verification of Proposition 7. This proposition can be reformulated as follows.

Corollary 4. *There is a well defined map from $\mathcal{L}(e)$ to the set of pairs $(e,\ C)$ where C is a conjugacy class in $A_k(e)$. This map is defined as follows. Assume that $((\mathfrak{g}_{\mathbf{C}})_J,\ \mathfrak{q},\ W) \in \mathcal{L}(e)$, $z \in Z_J \cap K_{\mathbf{C}}$, and $z(Z_J \cap K_{\mathbf{C}})_0$ generates $Z_J \cap K_{\mathbf{C}}/ (Z_J \cap K_{\mathbf{C}})_0$. Let $[((\mathfrak{g}_{\mathbf{C}})_J,\ \mathfrak{q},\ W)]$ denote the set of $K_{\mathbf{C}}$ conjugates of $((\mathfrak{g}_{\mathbf{C}})_J,\ \mathfrak{q},\ W)$ in $\mathcal{L}(e)$. Let $[z(K^e_{\mathbf{C}})_0]$ denote the conjugacy class of (the coset) $z(K^e_{\mathbf{C}})_0$ in $A_k(e)$.*

Suppose that the pair $((\mathfrak{g}_{\mathbf{C}})_J,\ e)$ is not one of those in group C in the proof of Proposition 7. Then we stipulate that

$$((\mathfrak{g}_{\mathbf{C}})_J,\ \mathfrak{q},\ W) \mapsto (e, [z({K_{\mathbf{C}}}^e)_0]). \tag{2.14}$$

Suppose $((\mathfrak{g}_{\mathbf{C}})_J,\ e)$ lies in group C. Let $((\mathfrak{g}_{\mathbf{C}})'_J,\ \mathfrak{q}',\ W')$ be a representative of the other $K_{\mathbf{C}}$ conjugacy class in $\mathcal{L}(e)$. Then each triple in $[((\mathfrak{g}_{\mathbf{C}})_J,\ \mathfrak{q},\ W)]$ maps to $(e, [z(K^e_{\mathbf{C}})_0])$ and each triple in $[((\mathfrak{g}_{\mathbf{C}})'_J,\ \mathfrak{q}',\ W')]$ maps to $(e, [z^3(K^e_{\mathbf{C}})_0])$.

If $z \in G_{\mathbf{C}}$, and $S \subset G_{\mathbf{C}}$. Let $Z_{\mathfrak{g}_{\mathbf{C}}}(x,\ S)$ (resp., $Z_{G_{\mathbf{C}}}(x,\ S)$) denote the centralizer of the set $\{z,\ S\}$ in $\mathfrak{g}_{\mathbf{C}}$ (resp. $G_{\mathbf{C}}$). We will require the following result which is a consequence of results about automorphisms of semisimple Lie algebras in [11] (See Problem 16 and Theorem 3 of chapter 4, section 4).

Proposition 8. [11] *Let R be a reductive subgroup of $G_{\mathbf{C}}$. Let $x,\ y \in R$ be two semisimple elements in R whose images in R/R_0 are in the same conjugacy class. Let S be a maximal torus in the reductive group $(R^x)_0$. Then for some $g \in R$, and $s \in S$, we have $gyg^{-1} = xs$. In particular, $Z_{\mathfrak{g}_{\mathbf{C}}}(x,\ S) \subset Ad(g)Z_{\mathfrak{g}_{\mathbf{C}}}(y)$.*

Definition 11. *Assume that $\mathfrak{g}$ is simple. Let L be an elliptic pseudo Levi subgroup with center Z and Lie algebra $\mathfrak{l}$. Given a nilpotent $e \in \mathfrak{p}_{\mathbf{C}}$ and a conjugacy class C in $A_k(e)$, we say that $\mathfrak{l}$ has the key property for $(e,\ C)$ if $e \in \mathfrak{l}$ and there exists $z \in Z \cap K_{\mathbf{C}}$ such that:*

(a) The image of z generates the cyclic group $Z \cap K_{\mathbf{C}}/(Z \cap K_{\mathbf{C}})_0$ and
(b) the image of z in $A_k(e)$ belongs to C.

Remark 7. *Assume that $\mathfrak{l}$ and z are as in Definition 11. Since $\mathfrak{l}$ is an elliptic, pseudo Levi subalgebra, $\mathfrak{l} = \mathfrak{g}_{\mathbf{C}}^{z'}$ for some $z' \in Z \cap K_{\mathbf{C}}$. Since the image of z generates $Z \cap K_{\mathbf{C}}/(Z \cap K_{\mathbf{C}})_0$ and this quotient is cyclic, we have $z' = z^n s$ for some non-negative integer n and $s \in (Z \cap K_{\mathbf{C}})_0$. Therefore,*

$$\mathfrak{l} \subset Z_{\mathfrak{g}_{\mathbf{C}}}(z,\ (Z\ \cap\ K_{\mathbf{C}})_0) \subset Z_{\mathfrak{g}_{\mathbf{C}}}(z^n s) = \mathfrak{l},$$

so that $\mathfrak{l} = Z_{\mathfrak{g}_{\mathbf{C}}}(z,\ (Z \cap K_{\mathbf{C}})_0)$.

Assume that $(e,\ C)$ is a pair as in Definition 11. We will construct an elliptic pseudo Levi subalgebra $\mathfrak{l}$ with the key property for $(e,\ C)$. We proceed as in Sommers [14]. Assume that $\mathfrak{m} = \{x,\ e,\ f\}$ is a normal triple containing e. Set $R = K_{\mathbf{C}}^{\mathfrak{m}}$. Since $A_k(e) = R/R_0$, , let $z \in R$ represent C. Since each element in R can be written as the product of a semi-simple and a unipotent element which commute, and the unipotent factor lies in R_0, we may assume that z is semisimple.

Lemma 11. *Let S be a maximal Cartan subgroup of $(R^z)_0$. Then,*

$$\mathfrak{l} := Z_{G_{\mathbf{C}}}(z,\ S) \tag{2.15}$$

is an elliptic pseudo-Levi subalgebra containing e. If $\mathfrak{s}$ is the Lie algebra of S, then $\mathfrak{s} = \mathfrak{s} \cap \mathfrak{k} \oplus i(\mathfrak{s} \cap \mathfrak{k})$. So that, $S = S \cap K \exp(i(\mathfrak{s} \cap \mathfrak{k}))$. If we choose a monothetic element s in the torus $S \cap K$, then L will be the connected component of $G_{\mathbf{C}}^{zs}$.

Proof. By construction, the element z satisfies part (b) of Definition 11. So it remains to verify part (a) of the definition.

If $\mathfrak{g}$ is equal rank, then $Z(L) = Z(L) \cap K_{\mathbf{C}}$, so Sommers' work implies that part (a) of Definition 11 holds.

If $\mathfrak{g}$ is not equal rank, then either (i) $\mathfrak{g}$ is classical; (ii) $\mathfrak{g} = EIV$; or (iii) $\mathfrak{g} = EI$.

Assume we are in case (i) or case (ii). By applying Proposition 4, we find that $Z(L) \cap K_{\mathbf{C}}/(Z(L) \cap K_{\mathbf{C}})_0$ is either trivial or $\mathbf{Z}_2$. From Sommers' results (see Remark 3 of [13]), we know that the image of z generates $Z(L)/Z(L)_0$. If $Z(L)/Z(L)_0$ is not trivial, then $Z(L) \cap K_{\mathbf{C}}/(Z(L)_0 \cap K_{\mathbf{C}})$ is non trivial and must be generated by the image of z. Recall the exact sequence (2.5):

$$(1) \to \frac{Z(L)_0 \cap K_{\mathbf{C}}}{(Z(L) \cap K_{\mathbf{C}})_0} \to \frac{Z(L) \cap K_{\mathbf{C}}}{(Z(L) \cap K_{\mathbf{C}})_0} \to \frac{Z(L) \cap K_{\mathbf{C}}}{Z(L)_0 \cap K_{\mathbf{C}}} \to (1).$$

It follows that $Z(L) \cap K_{\mathbf{C}}/(Z(L) \cap K_{\mathbf{C}})_0$ is non trivial. Hence it is equal to $\mathbf{Z}_2$ and is generated by the image of z.

On the other hand, if $Z(L)/Z(L)_0$ is trivial, then either

(1) $Z(L)_0 \cap K_{\mathbf{C}}/(Z(L) \cap K_{\mathbf{C}})_0$ is trivial or,

(2) $Z(L)_0 \cap K_{\mathbf{C}}/(Z(L) \cap K_{\mathbf{C}})_0 = \mathbf{Z}_2$

by Propositions 4 and 5. In case (1) the image of z is the trivial coset and there is nothing to prove. In subcase (2) , $z \notin (Z(L) \cap K_{\mathbf{C}})_0$ since it was chosen to lie outside of R_0. So the image of z must generate the whole quotient.

In case (iii) the only elliptic pseudo Levi subalgebras where the cardinality of $Z(L) \cap K_{\mathbf{C}}/(Z(L) \cap K_{\mathbf{C}})_0$ exceeds 2 are those whose real forms have semi-simple parts of isomorphism type $su(2,\ 1) \oplus sl(3, \mathbf{C})$ and $su(2,\ 2) \oplus sl(2,\ \mathbf{C})$. For the first subalgebra, $Z(L) \cap K_{\mathbf{C}}\ /\ (Z(L) \cap K_{\mathbf{C}})_0$ is $\mathbf{Z}_3$ and since $z \notin R_0$, $z(Z(L) \cap K_{\mathbf{C}})_0$ must generate the full quotient. For the second subalgebra, $Z(L) \cap K_{\mathbf{C}}/(Z(L) \cap K_{\mathbf{C}})_0$ is $\mathbf{Z}_4$. The exact sequence (2.5) becomes:

$$(1) \to \mathbf{Z}_2 \to \mathbf{Z}_4 \to \mathbf{Z}_2 \to (1). \tag{2.16}$$

The coset $z(Z(L) \cap K_{\mathbf{C}})_0$ must generate $Z(L) \cap K_{\mathbf{C}}/(Z(L)_0 \cap K_{\mathbf{C}})$, the final $\mathbf{Z}_2$ in the preceding exact sequence. Thus $z \notin Z(L)_0 \cap K_{\mathbf{C}}$, and so the coset $z(Z(L) \cap K_{\mathbf{C}})_0$ does not have order 2 in $Z(L) \cap K_{\mathbf{C}}/(Z(L) \cap K_{\mathbf{C}})_0$. Therefore, this coset must have order 4 and generate all of $Z(L) \cap K_{\mathbf{C}}/(Z(L) \cap K_{\mathbf{C}})_0 = \mathbf{Z}_4$.

It follows that in all cases, L has the key property for $(e,\ C)$.

Proposition 9. *The subalgebra $\mathfrak{l}$ defined in (2.15) is minimal among the elliptic pseudo-levi suablgebras with the key property for $(e,\ C)$. Moreover any other minimal pseudo-Levi subalgebra with the key property for $(e,\ C)$ is conjugate to $\mathfrak{l}$ by an element of $A_k(e)$.*

Proof. We imitate the proof of Proposition 12 in [14]. We use the notation immediately preceding equation (2.15).

Suppose $\mathfrak{l}'$ is another elliptic pseudo-Levi subalgebra with the key property for the pair $(e,\ C)$. Let L' be the corresponding connected subgroup of $G_{\mathbf{C}}$. Since $\mathfrak{l}'$ has the key property for $(e,\ C)$, Remark 7 implies that there exists $z' \in Z(L') \cap K_{\mathbf{C}}$ and $s' \in (Z(L') \cap K_{\mathbf{C}})_0$ such that $\mathfrak{l}' = \mathfrak{g}_{\mathbf{C}}^{z's'}$ and the image of z' in $A_k(e)$ lies in C. It follows that the image of $z's'$ in $A_k(e)$ also lies in C.

Let $\mathfrak{m} = \{x',\ e,\ f'\}$ be a normal triple in $\mathfrak{l}'$. Then $z's'$ centralizes $\mathfrak{m}'$. By the work of Kostant and Rallis [9], there exists $g \in {K_{\mathbf{C}}}^e$ such that $Ad(g)(\mathfrak{m}') = \mathfrak{m}$. If we conjugate $\mathfrak{l}'$, L', and $z's'$ by g, we can assume that z and $z's'$ are semisimple elements in R which represent the same conjugacy class in the component group of R. Applying Proposition 8, there exists $g' \in R$ such that $Z_{\mathfrak{g}_{\mathbf{C}}}(z,\ S) = \mathfrak{l}$ is contained in $Ad(g')({\mathfrak{g}_{\mathbf{C}}}^{z's'}) = Ad(g')\mathfrak{l}'$.

Proposition 10. *Let $\mathfrak{l}$ be a minimal elliptic pseudo-Levi subalgebra with the key property for $(e,\ C)$. Then, e is noticed in $\mathfrak{l}$.*

Proof. We adapt the proof of Proposition 13 in [14].

Let $z(\mathfrak{l})$ be the center of $\mathfrak{l}$ and $\mathfrak{c} = z(\mathfrak{l}) \cap \mathfrak{k}_{\mathbf{C}}$. Suppose that e is not noticed in $\mathfrak{l}$. Let $y \in \mathfrak{l} \cap \mathfrak{k}_{\mathbf{C}}$ be a semi-simple element such that $[y,\ e] = 0$ but $y \notin \mathfrak{c}$. Set $\mathfrak{c}^*$ equal to the subalgebra generated by $\mathfrak{c} \cup \{y\}$. Then $\mathfrak{c}^*$ is a toral subalgebra and $\mathfrak{l}^* = \mathfrak{l}^{\mathfrak{c} \cup \{y\}} = \mathfrak{l}^{\mathfrak{c}^*}$ is an elliptic pseudo-Levi subalgebra which contains e. L^* denotes the connected subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{l}^*$.

We now show that $\mathfrak{l}^*$ has the key property for $(e,\ C)$. Let z be an element satisfying (a) and (b) of Definition 11 for $\mathfrak{l}$. Then the image of z in $A_k(e)$ lies in C and by Remark 7, $L = Z_{G_{\mathbf{C}}}(z,\ (Z(L) \cap K_{\mathbf{C}})_0)_0$. Since $L^* = L^{\mathfrak{c}^*}$, and $Z(L)$ fixes each element of $\mathfrak{c}^*$, we have $Z(L) \subset Z(L^*)$, so that $z \in Z(L^*) \cap K_{\mathbf{C}}$. It remains to show that the image of z in $Z(L^*) \cap K_{\mathbf{C}}/(Z(L^*) \cap K_{\mathbf{C}})_0$ generates this cyclic group.

It follows from the definition of $\mathfrak{l}^*$ that $L^* = Z_{G_{\mathbf{C}}}(z,\ \exp(\mathfrak{c}^*))_0$. Moreover $\mathfrak{c}^* \subset z(\mathfrak{l}^*) \cap \mathfrak{k}_{\mathbf{C}}$. Remark 3 in [13] implies that the image of z generates the cyclic group $Z(L^*)/Z(L^*)_0 = Z(L^*) \cap K_{\mathbf{C}}/Z(L^*)_0 \cap K_{\mathbf{C}}$. Now consider the exact sequence (2.5):

$$(1) \to \frac{Z(L^*)_0 \cap K_{\mathbf{C}}}{(Z(L^*) \cap K_{\mathbf{C}})_0} \to \frac{Z(L^*) \cap K_{\mathbf{C}}}{(Z(L^*) \cap K_{\mathbf{C}})_0} \to \frac{Z(L^*) \cap K_{\mathbf{C}}}{Z(L^*)_0 \cap K_{\mathbf{C}}} \to (1).$$

If $\mathfrak{g}$ is equal rank, then $Z(L^*) \cap K_{\mathbf{C}} = Z(L^*)$, so that $Z(L^*) \cap K_{\mathbf{C}} / Z(L^*)_0 \cap K_{\mathbf{C}} = Z(L^*) \cap K_{\mathbf{C}} / (Z(L^*) \cap K_{\mathbf{C}})_0$ and we are done.

If $\mathfrak{g}$ is not equal rank, then we only have to consider the case in which

$$Z(L^*)_0 \cap K_{\mathbf{C}} / (Z(L^*) \cap K_{\mathbf{C}})_0 = \mathbf{Z}_2.$$

(Otherwise by part (b) of Lemma 4, $Z(L^*)_0 \cap K_{\mathbf{C}} / (Z(L^*) \cap K_{\mathbf{C}})_0$ is trivial and again $Z(L^*) \cap K_{\mathbf{C}} / Z(L^*)_0 \cap K_{\mathbf{C}} = Z(L^*) \cap K_{\mathbf{C}} / (Z(L^*) \cap K_{\mathbf{C}})_0$.) In this case, we may also assume that $Z(L^*) \cap K_{\mathbf{C}} / Z(L^*)_0 \cap K_{\mathbf{C}}$ is not trivial. For if this quotient is trivial, $Z(L^*) \cap K_{\mathbf{C}} / (Z(L^*) \cap K_{\mathbf{C}})_0 = \mathbf{Z}_2$, which then must be generated by the image of z. We are thus reduced to the case in which $\mathfrak{g} = EI$, and the semi-simple part of $\mathfrak{l}^*_{\mathbf{R}} = su(2,\ 2) \oplus sl(2,\ \mathbf{C})$. In the proof of Lemma 11, we showed that image of z generated $Z(L^*) \cap K_{\mathbf{C}} / (Z(L^*) \cap K_{\mathbf{C}})_0$.

We have now established that in all cases, $\mathfrak{l}^*$ has the key property for $(e,\ C)$. This contradicts the minimality of $\mathfrak{l}$. Therefore, the element y postulated at the beginning of the proof does not exists. Thus e must be noticed in $\mathfrak{l}$.

Proof of Theorem 1.

By Corollary 4 there is a well defined map from $\mathcal{L}(e)$ to the set of pairs $(e,\ C)$ where C is a conjugacy class in $A_k(e)$. The assignment is surjective because given a pair $(e,\ C)$ we can first construct (as in equation (2.15)) an elliptic pseudo Levi subalgebra $\mathfrak{l}$ which contains e and has the key property for $(e,\ C)$. Propositions 9 and 10 then imply that e is noticed in $\mathfrak{l}$. We then obtain the parabolic $\mathfrak{q}$, and the subspace W to complete the triple $(\mathfrak{l},\ \mathfrak{q},\ W)$.

It remains to show that the mapping is injective on the $K_{\mathbf{C}}$ conjugacy classes of $\mathcal{L}(e)$. Suppose that $(\mathfrak{l},\ \mathfrak{q},\ W)$ and $(\mathfrak{l}',\ \mathfrak{q}',\ W')$ are two triples in $\mathcal{L}(e)$ that are mapped to the pair $(e,\ C)$. Then $\mathfrak{l}$ and $\mathfrak{l}'$ must have the key property for the pair $(e,\ C)$. Since e is noticed in $\mathfrak{l}$ and $\mathfrak{l}'$, each pseudo Levi subalgebra must be minimal with respect to having the key property. (Here is an explanation of this fact. Assume that $\mathfrak{l}^* \subset \mathfrak{l}$ is an elliptic pseudo Levi subalgebra with the key property for $(e,\ C)$. We must have $\mathfrak{l}^* = Z_{\mathfrak{g}_{\mathbf{C}}}(z,\ (Z(L^*) \cap K_{\mathbf{C}})_0)$. If $\mathfrak{l}^*$ is a proper subalgebra of $\mathfrak{l}$, then $(Z(L) \cap K_{\mathbf{C}})_0$ must be a proper subset of $(Z(L^*) \cap K_{\mathbf{C}})_0$, contradicting the fact that e is noticed in $\mathfrak{l}$.)

Let $\{x,\ e,\ f\}$ and $\{x',\ e,\ f'\}$ be normal $sl(2)$-triples in $\mathfrak{l}$ and $\mathfrak{l}'$ respectively. By the proof of Proposition 9 there exists $k \in {K_{\mathbf{C}}}^{\{x,\ e,\ f\}}$ such that $k \cdot \mathfrak{l} = \mathfrak{l}'$. Therefore, the triple $\{x,\ e,\ f\}$ also belongs to $\mathfrak{l}'$. We can now find $k^* \in (L' \cap K_{\mathbf{C}})^e$ so that $x' = k^* x$. Since $\mathfrak{q}'$ (respectively $k \cdot \mathfrak{q}$) is the sum of the non negative eigenspaces of x' (respectively x) in $\mathfrak{l}'$, and W' (respectively kW) is the 2-eigenspace of x' (respectively x) intersected with $\mathfrak{p}_{\mathbf{C}}$ we see that the triples $(\mathfrak{l},\ \mathfrak{q},\ W)$ and $(\mathfrak{l}',\ \mathfrak{q}',\ W')$ are conjugate under the element $k^* k$.

Proof of Theorem 2.

Since e is noticed in $\mathfrak{g}_{\mathbf{C}}$, $K^{\{x,\ e,\ f\}}$ is finite. Thus $A_k(e) = K_{\mathbf{C}}^{\{x,\ e,\ f\}} = K^{\{x,\ e,\ f\}}$. Since e is distinguished in $\widetilde{\mathfrak{g}_{\mathbf{C}}}(e)$, $\widetilde{A}(e)$, the component group of e in $\widetilde{G_{\mathbf{C}}}(e)$ is the same as $\widetilde{G_{\mathbf{C}}}(e)^{\{x,\ e,\ f\}}$. We also have $K_{\mathbf{C}}^{\{x,\ e,\ f\}} \subset K_{\mathbf{C}}^x \subset \widetilde{G_{\mathbf{C}}}(e)$. In addition, $\widetilde{G_{\mathbf{C}}}(e)^{\{x,\ e,\ f\}} \subset K_{\mathbf{C}}^x$. Therefore, $\widetilde{A}(e) = \widetilde{G_{\mathbf{C}}}(e)^{\{x,\ e,\ f\}} = K_{\mathbf{C}}^{\{x,\ e,\ f\}} = K^{\{x,\ e,\ f\}} = A_k(e)$.

Let $\widetilde{\mathcal{L}}(e)$ denote the set of pairs $(\tilde{\mathfrak{l}}',\ \tilde{\mathfrak{q}}')$ such that $\tilde{\mathfrak{l}}'$ is a pseudo elliptic Levi subalgebra of $\widetilde{\mathfrak{g}_{\mathbf{C}}}(e)$ and $\tilde{\mathfrak{q}}'$ is a distinguished parabolic of $\tilde{\mathfrak{l}}'$ for e. Suppose $(\mathfrak{l},\ \mathfrak{q},\ W) \in$

$\mathcal{L}(e)$. If $\widetilde{\mathfrak{l}}$ is the r-carrier subalgebra of e in $\mathfrak{l}$, then e is distinguished in $\widetilde{\mathfrak{l}}$, and so lies in a distinguished parabolic subalgebra $\widetilde{\mathfrak{q}}$ of $\widetilde{\mathfrak{l}}$. The triple $(\mathfrak{l},\ \mathfrak{q},\ W)$ is not one of those in group C in the proof of Proposition 7. (Otherwise the hypotheses on $\widetilde{G}_{\mathbf{C}}(e)$ would be violated.) This implies that there exists a cyclic generator $z \in Z(L) \cap K_{\mathbb{C}}$ such that $\mathfrak{l} = \mathfrak{g}_{\mathbb{C}}^z$. (If z' is another such generator, then z' is conjugate to z under $K_{\mathbb{C}}^{\{x,\ e,\ f\}}$.) Then, $\widetilde{\mathfrak{l}} = \widetilde{\mathfrak{g}_{\mathbb{C}}}(e)^z$ showing that $\widetilde{\mathfrak{l}}$ is a pseudo Levi subalgebra of $\widetilde{\mathfrak{g}_{\mathbb{C}}}(e)$. Therefore, the assignment

$$(\mathfrak{l},\ \mathfrak{q},\ W) \mapsto (\widetilde{\mathfrak{l}},\ \widetilde{\mathfrak{q}}) \tag{2.17}$$

gives a well defined map from $K_{\mathbb{C}}$ conjugacy classes of triples in $\mathcal{L}(e)$ to $\widetilde{G_{\mathbb{C}}}(e)$ conjugacy classes of pairs in $\widetilde{\mathcal{L}}(e)$.

It is not hard to show that the mapping in (2.17) is injective. Since $A_k(e) = \widetilde{A}(e)$, the two component groups have the same conjugacy classes. The surjectivity of (2.17) follows from Theorem 1 and Theorem 14 of [14].

Appendix A. Simple root systems for real exceptional simple algebras

We list certain simple root systems (of the real exceptional simple Lie algebras) on the fundamental Cartan subalgebra that are needed in our proofs. For each simple algebra below, π_0 denotes the Vogan system of simple roots described by Knapp in Appendix C of [7]. We also choose the same underlying positive root system for $(\mathfrak{k}_{\mathbb{C}},\ \mathfrak{t}_{\mathbb{C}})$ as Knapp. We call this positive root system Δ_k^+ and note that π_0 is compatible with Δ_k^+.

EI. We have $\pi_0 = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$. Here $\alpha_1,\ \alpha_3,\ \alpha_5,\ \alpha_6$ are complex with $\theta(\alpha_1) = \alpha_6$, and $\theta(\alpha_3) = \alpha_5$. α_2 is noncompact imaginary. α_4 is compact imaginary. Set

$$\pi_I = s_{\alpha_2}\pi_0 = \{\alpha_1,\ -\alpha_2,\ \alpha_3,\ \alpha_2+\alpha_4,\ \alpha_5,\ \alpha_6\}.$$

EII. $\pi_0 = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$. Here, α_i is compact if $i \neq 2$, and α_2 is noncompact.

$$\pi_I = \{\beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \beta_5,\ \beta_6\},$$

where $\beta_1 = -(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$, $\beta_2 = \alpha_2+\alpha_3+2\alpha_4+\alpha_5$, $\beta_3 = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$, $\beta_4 = -(\alpha_2+\alpha_3+\alpha_4+\alpha_5)$, $\beta_5 = \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$ and $\beta_6 = -(\alpha_2+\alpha_4+\alpha_5+\alpha_6)$.

EV. $\pi_0 = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7\}$. Here, α_i is compact if $i \neq 2$, and α_2 is noncompact.

$$\pi_I = \{\beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \beta_5,\ \beta_6,\ \beta_7\},$$
$$\pi_{II} = s_{\beta_7}s_{\beta_5}s_{\beta_2}\pi_I = \{\beta_1,\ -\beta_2,\ \beta_3,\ \beta_2+\beta_4+\beta_5,\ -\beta_5,\ \beta_5+\beta_6+\beta_7,\ -\beta_7\}$$

where $\beta_1 = -(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$, $\beta_2 = \alpha_2+\alpha_3+2\alpha_4+\alpha_5$, $\beta_3 = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$, $\beta_4 = -(\alpha_2+\alpha_3+\alpha_4+\alpha_5)$, $\beta_5 = \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$, $\beta_6 = -(\alpha_2+\alpha_4+\alpha_5+\alpha_6)$, and $\beta_7 = \alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7$. Note that ψ_I, the longest positive root of the system determined by π_I, is given by $\psi_I = \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$.

EVI. $\pi_0 = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7\}$. Here, α_i is compact if $i \neq 1$, and α_1 is noncompact.

$$\pi_I = \{\beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \beta_5,\ \beta_6,\ \beta_7\},$$

where $\beta_1 = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$, $\beta_2 = \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$, $\beta_3 = \alpha_6$, $\beta_4 = -(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6)$, $\beta_5 = \alpha_4$, $\beta_6 = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7$, and $\beta_7 = -(\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7)$.

EVIII. $\pi_0 = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7, \alpha_8\}$. Here, α_i is compact if $i \neq 1$, and α_1 is noncompact.

$$\pi_I = \{\beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \beta_5,\ \beta_6,\ \beta_7, \beta_8\},$$

where $\beta_1 = -(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5)$, $\beta_2 = \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$, $\beta_3 = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6$, $\beta_4 = -(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6)$, $\beta_5 = \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7$, $\beta_6 = -(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7)$, $\beta_7 = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$, and $\beta_8 = -(\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8)$.

EIX. $\pi_0 = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7, \alpha_8\}$. Here, α_i is compact if $i \neq 8$, and α_8 is noncompact.

$$\pi_I = \{\beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \beta_5,\ \beta_6,\ \beta_7, \beta_8\},$$

where $\beta_1 = \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8$, $\beta_2 = \alpha_5$, $\beta_3 = -(\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8)$, $\beta_4 = \alpha_4$, $\beta_5 = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8$, $\beta_6 = -(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8)$, $\beta_7 = \alpha_1$, and $\beta_8 = \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8$.

FI. $\pi_0 = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}$ where α_1, α_2, and α_3 are compact and α_4 is noncompact. α_1 and α_2 are short. α_3 and α_4 are long.

$$\pi_I = \{\beta_1,\ \beta_2,\ \beta_3,\ \beta_4\}$$

where $\beta_1 = -(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$, $\beta_2 = \alpha_1+2\alpha_2+\alpha_3+\alpha_4$, $\beta_3 = -(2\alpha_2+\alpha_3+\alpha_4)$, and $\beta_1 = 2\alpha_2+2\alpha_3+\alpha_4$.

G. $\pi_0 = \{\alpha,\ \beta\}$ where α is compact and short, and β is noncompact and long.

$$\pi_I = \{\alpha+\beta,\ -\beta\}$$

Appendix B. $K_{\mathbb{C}}$ conjugacy classes of elliptic pseudo Levi subalgebras for real exceptional simple algebras

The authors are not aware of any published classification of the $K_{\mathbb{C}}$ conjugacy classes of elliptic pseudo Levi subalgebras for a simple exceptional real algebra $\mathfrak{g}$. So we will briefly discuss how one can establish the assertions made in the proof of Proposition 7 concerning $K_{\mathbb{C}}$ conjugacy classes of various isomorphism types.

For specificity, we assume $\mathfrak{g} = EV$ and consider the $K_{\mathbb{C}}$ conjugacy classes of elliptic pseudo Levi subalgebras $\mathfrak{l}$ such that $\mathfrak{l}_{\mathbf{R}}$ is of isomorphism type $su(3,\ 3) \oplus su(2,\ 1)$ or of type $2su(2,\ 2) \oplus sl(2,\ \mathbf{R})$. Given $\mathfrak{l}_{\mathbf{R}}$, one can find a positive system of roots for $(\mathfrak{k}_{\mathbb{C}},\ \mathfrak{t}_{\mathbb{C}})$, a compatible set of simple roots π for $\mathfrak{g}_{\mathbb{C}}$, and a subset $J \subsetneq \widetilde{\pi}$ such that $\mathfrak{l}_{\mathbf{R}} = \mathfrak{g}_J$. Assume that the positive system for $(\mathfrak{k}_{\mathbb{C}},\ \mathfrak{t}_{\mathbb{C}})$ is Δ_k^+, the same one chosen in Appendix A. Since $\mathfrak{l}_{\mathbf{R}}$ is quasisplit, there exists $w \in \mathbf{W}_J$ such that all the roots in $w(J)$ are noncompact. We have $w(J) \subsetneq \widetilde{w(\pi)}$. But $w(\pi)$ is conjugate under $\mathbf{W}_k$ to a set of simple roots π' which is compatible with Δ_k^+. So we might as well assume from the start that J is a special simple root system for $\mathfrak{g}_J$. Let $\mathcal{S}$

denote the set of simple sets π' for EV which are compatible with Δ_k^+. There are exactly 72 simple systems π' in $\mathcal{S}$, and no pair of these systems is conjugate under $\mathbf{W}_k$. By inspecting each of the corresponding $\widetilde{\pi}'$, we find that the following holds.

Set ψ_I equal to the longest positive root relative to the simple system π_I. If $\mathfrak{g}_J$ is of type $su(3,\ 3)\ \oplus\ su(2,\ 1)$, then $\pi = \pi_I$ or π_{II}. In fact, J must be one of the following subsets:

$$\begin{aligned}
J_1 &= \{-\psi_I,\ \beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \beta_6,\ \beta_7\},\\
J_2 &= \{-\psi_I,\ \beta_1,\ \beta_2,\ \beta_4,\ \beta_5,\ \beta_6,\ \beta_7\},\\
J_3 &= \{-\psi_I,\ \beta_1,\ -\beta_2,\ \beta_3,\ \beta_2+\beta_4+\beta_5,\ \beta_5+\beta_6+\beta_7,\ -\beta_7\}\\
J_4 &= \{-\psi_I,\ \beta_1,\ -\beta_2,\ \beta_2+\beta_4+\beta_5,\ -\beta_5,\ \beta_5+\beta_6+\beta_7,\ -\beta_7\}.
\end{aligned}$$

We observe that $J_4 = s_{\beta_2}s_{\beta_5}s_{\beta_7}(J_2)$. Since $\beta_2,\ \beta_5,\ \beta_7 \in J_2$, we see that J_4 and J_2 determine (span) the same root system, and hence determine the same algebra $\mathfrak{g}_J$. Furthermore, there exists $w^\flat \in \mathbf{W}_k$ such that $w(J_1) = J_4$ and $w(J_2) = J_3$. This element is defined as follows: $-\psi_I \mapsto -\beta_7$, $\beta_1 \mapsto \beta_5+\beta_6+\beta_7$, $\beta_3 \mapsto -\beta_5$, $\beta_4 \mapsto \beta_2+\beta_4+\beta_5$, $\beta_2 \mapsto -\beta_2$, $\beta_6 \mapsto \beta_1$ and $\beta_7 \mapsto -\psi_I$. (The fact that $w^\flat \in \mathbf{W}_k$ can be demonstrated as follows. The roots of the simple system π_0 of E_7 can be rewritten in terms of a certain dependent set of vectors ε_i, $i = 1, \dots, 8$. (See Table 1 in Reference Chapter of [11].) $\mathbf{W}_k$ can be identified with the set of permutations of the ε_i, and $w^\flat$ can be shown to be such a permutation.) It follows from Proposition 2 that the subalgebras $\mathfrak{g}_{J_i}$, $i = 1, \dots, 4$ are conjugate under $K_{\mathbb{C}}$. So there is a single $K_{\mathbb{C}}$ conjugacy class of isomorphism type $su(3,\ 3) \oplus su(2,\ 1)$.

If $\mathfrak{g}_J$ is of type $2su(2,\ 2) \oplus sl(2,\ \mathbf{R})$, then (a) $\pi = \pi_I$ or π_{II} or (b) there exists a noncompact $\alpha \in J \cap \pi$ such that $s_\alpha(\pi) = \pi_I$ or π_{II}. In fact, J must be one of the following subsets:

$$\begin{aligned}
J_1' &= \{-\psi_I,\ \beta_1,\ \beta_2,\ \beta_3,\ \beta_5,\ \beta_6,\ \beta_7\},\\
J_2' &= \{-\psi_I,\ \beta_1,\ -\beta_2,\ \beta_3,\ \beta_5,\ \beta_6,\ \beta_7\},\\
J_3' &= \{-\psi_I,\ \beta_1,\ -\beta_2,\ \beta_3,\ -\beta_5,\ \beta_5+\beta_6+\beta_7,\ -\beta_7\}\\
J_4' &= \{-\psi_I,\ \beta_1,\ \beta_2,\ \beta_3,\ -\beta_5,\ \beta_5+\beta_6+\beta_7,\ -\beta_7\}.
\end{aligned}$$

We see that $J_2' = s_{\beta_2}(J_1')$, $J_3' = s_{\beta_2}s_{\beta_5}s_{\beta_7}(J_1')$, and $J_4' = s_{\beta_5}s_{\beta_7}(J_1')$. Since $\beta_2,\ \beta_5,\ \beta_7 \in J_1'$, the subsets J_i' span the same root system, and so determine the same algebra $\mathfrak{g}_J$. It follows that there is a single $K_{\mathbb{C}}$ conjugacy class of isomorphism type $2su(2,\ 2) \oplus sl(2,\ \mathbf{R})$.

The other assertions in the proof of Proposition 7 about $K_{\mathbb{C}}$ conjugacy classes of elliptic pseudo Levi subalgebras of other simple $\mathfrak{g}$ can be established by arguments similar to those above utilizing the simple root systems π_I defined in Appendix A.

References

[1] D. Z. Djokovic. Classification of nilpotent elements in simple exceptional real Lie algebras of inner type and description of their centralizers. *J. Alg.*, 112:503–524, 1988.

[2] D. Z. Djokovic. Classification of nilpotent elements in simple real Lie algebras $e_{6(6)}$ and $e_{6(-26)}$ and description of their centralizers. *J. Alg.*, 116:196–207, 1988.

[3] David A. Vogan Jr. Gelfand-Kirillov dimension for Harish-Chandra modules. *Inventiones math*, 48:75–98, 1978.

[4] David A. Vogan Jr. The method of coadjoint orbits for real reductive groups. In *IAS/Park City Mathematics Series*, volume 6, 1998.

[5] Noriaki Kawanaka. Orbits and stabilizers of nilpotent elements of a graded semisimple Lie algebra. *J. Fac. Sci. Univ. Tokyo Sect IA Math.*, 34(3):573–597, 1987.

[6] Donald R. King. The component groups of nilpotents in exceptional simple real lie algebras. *Communications in Algebra*, 20(1):219–284, 1992.

[7] Anthony W. Knapp. *Lie Groups Beyond an Introduction*, volume 140 of *Progress in Mathematics*. Birkhauser, Boston, 1996.

[8] Anthony W. Knapp and Jr. David A. Vogan. *Cohomological Induction and Unitary Representations*, volume 45 of *Princeton Mathematics Series*. Princeton University Press, Princeton, New Jersey, 1995.

[9] B. Kostant and S. Rallis. Orbits and representations associated with symmetric spaces. *Amer. J. Math.*, 93:753–809, 1971.

[10] Alfred G. Noel. Nilpotent orbits and θ-stable parabolic subalgebras. *Representation Theory*, 2:1–32, 1998.

[11] A. L. Onishchik and E. B. Vinberg. *Lie Groups and Algebraic Groups*. Springer, Berlin, 1990.

[12] J. Sekiguchi. Remarks on real nilpotent orbits of a symmetric pair. *J. Math. Soc. Japan*, 39(1):127–138, 1987.

[13] Eric Sommers. A generalization of the Bala-Carter theorem for nilpotent orbits. *International Mathematics Research Notices*, 11:539–562, 1998.

[14] Eric N. Sommers. *Nilpotent Orbits and the Affine Flag Manifold*. PhD thesis, Massachusetts Institute of Technology, Sept 1997.

[15] E. B. Vinberg. On the classification of the nilpotent elements of graded Lie algebras. *Soviet Math. Dokl.*, 16(6):1517–1520, 1975.

[16] E. B. Vinberg. Classification of homogeneous nilpotent elements of a semisimple graded Lie algebra. *Selecta Mathematica Sovietica*, 6(1):15–35, 1987.