Quantization in Physics and Mathematics

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• There is a lot of symmetry in Nature.

• Mathematicians and scientists often used **GROUP THEORY** to study symmetry that is expressed by group transformations preserving some structure.

• Felix Klein (Das Erlanger Programm, 1872)

• Sophus Lie und Friedrich Engel
(Theorie derTransformationsgruppen, 1888-1893)

• WHAT IS A GROUP?

• A group $\mathfrak{G}$ is a set with an operation **like** Addition or Multiplication in the set of all real numbers.

Example

The set of permutations of 3 objects. Here the operation is the process that takes you from one permutation to another.

$$\mathfrak{S}_3 = \{\{a, b, c\}, \{a, c, b\}, \{b, a, c\}, \{b, c, a\}, \{c, a, b\}, \{c, b, a\}\}$$

GROUP REPRESENTATION

A representation of a group is a process that associates a matrix to each element of the group.

$$\{a, b, c\} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \{a, c, b\} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\{b, a, c\} \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \{b, c, a\} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\{c, a, b\} \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \{c, b, a\} \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

These matrices form a group $\mathfrak{G}$ under matrix Multiplication and we may think of $\mathfrak{G}$ **acting** on three dimensional vectors by permuting their components.

APPLICATION: PARITY IN QUANTUM MECHANICS

$\mathfrak{G} = \{id, r\}$ is defined by

$\times$	id	r
id	id	r
r	r	id

$\mathfrak{G}$ has exactly two irreducible representations

- Trivial representation: $id \to 1$, $r \to 1$
- Parity representation: $id \to 1$, $r \to -1$

Any other representation of $\mathfrak{G}$ must be a combination of these.

From nonrelativistic quantum mechanics in one dimension, a particle in a potential symmetric about $x = 0$ has energy eigenfunctions that are either symmetric if x is replaced by $-x$ (Trivial Representation) or antisymmetric corresponding (Parity Representation).

In general we can always choose the energy eigenstates to transform like irreducible representations of the group. See Howard Georgi's book Lie Algebras in Particle Physics

THE FUNDAMENTAL PROBLEM

Given a group $\mathfrak{F}$ find all the different representations of $\mathfrak{F}$

The problem is basically solved for :

•$\mathfrak{F}$ finite

•$\mathfrak{F}$ Reductive complex Lie groups: The set of all $n \times n$ invertible complex matrices for example.

If $\mathfrak{F}$ is a real reductive Lie group for example the set of orthogonal matrices with real entries then we do not have a complete solution yet. **THIS IS MY AREA OF RESEARCH.**

THE ORBIT OF A POINT UNDER A GROUP ACTION

Let SO_3 be the real group of rotations in the 3-dimensional space. If we fix a system of coordinates then the group rotates each point P in all possible ways. if P is not the origin the resulting **orbit** of P is a **sphere**. Such orbits are called HOMOGENEOUS SPACES. See figure:

GEOMETRY OF NILPOTENT ORBITS OF MATRICES

A non zero matrix A is **nilpotent** if $A^k = 0$ for some positive integer k.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad k = 2.$$

Let $\mathfrak{g}$ be the set of real matrices of the form $\begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}$

Description of $\mathfrak{N}$ the set of nilpotent matrices of $\mathfrak{g}$

Observe that $\begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}^{2k} = (x^2 + y^2 - z^2)^k * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Hence $\mathfrak{N}$ is the set of points (x, y, z) such that $x^2 + y^2 - z^2 = 0$. In other words $\mathfrak{N}$ is a double sheeted-cone. See figure:

DESCRIPTION OF NILPOTENT ORBITS OF $\mathfrak{g}$

Let $\mathfrak{G}$ be the group of matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$. Then $\mathfrak{G}$ acts on $\mathfrak{g}$ by conjugation $\mathfrak{G}\mathfrak{g}\mathfrak{G}^{-1}$. And $\mathfrak{G}$ admits exactly 3 nilpotent orbits on $\mathfrak{g}$. **It is a fact that for a real reductive Lie group the number of nilpotent orbits under conjugation is finite.**

Class I: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow$

Class II: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow$

Class III: $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Longrightarrow$

THE "ORBIT METHOD" PHILOSOPHY

Originally proposed by Alexander Kirillov in the 1960's.
But comes from Physics.

Pursued by Kostant and Auslander (1970's) Duflo in 1980's, Vogan and his school in 1990's.

We want to find all the represensions of $\boldsymbol{\mathfrak{G}}$

In many cases the method points us to new representations

Quantization

Physics	**Representation Theory**
(Phase Space)	(Symplectic Homogeneous Space)
$\parallel$	$\parallel$
Classical System	Admissible Nilpotent Orbit
$\Downarrow$	$\Downarrow$
QM System (Hilbert Space)	Irreducible Unitary Representation of $\boldsymbol{\mathfrak{G}}$
$\parallel$	$\parallel$
(Unitary Operators)	(Unitary Operators)

CLASSIFICATION OF ADMISSIBLE NILPOTENT ORBITS

Classical Reductive Real Lie Groups

- J. Schwartz (1987)
- T. Ohta (1991)

Exceptional Reductive Real Lie Groups

- A. Nœl (2002)

FIN DE L' EPISODE

A SUIVRE