From seen to unseen Lagrangians via algebraic geometry

Tim Perutz (University of Texas at Austin)
Joint work with Nick Sheridan (Princeton/IAS)

AMS Special session: Advances in symplectic geometry and topology
Mathematical Congress of the Americas, Guanajuato, Mexico

August 6, 2013
Calabi–Yau (CY) mirror symmetry

- A **CY variety** will mean a smooth, proper algebraic variety $X$ over a field, together with a volume form $\Omega$, that is, a trivialization of the canonical line bundle $\mathcal{K}_X$.
- Roughly (and incorrectly), mirror symmetry is an involution $(X, \Omega; \omega + ib) \leftrightarrow (\check{X}, \check{\Omega}; \check{\omega} + i\check{b})$
  on CY varieties $(X, \Omega)$ over $\mathbb{C}$ equipped with complexified Kähler forms $\omega + ib$ on $X^{an}$.
- Deformations of the class $[\omega + ib] \in H^{1,1}(X)$ correspond to first-order deformations of the variety $\check{X}$ in $H^1(T\check{X})$.
- More accurately (if incompletely), mirror symmetry is an involution on **maximally degenerating 1-parameter families of projective CY varieties**.
  Complexified Kähler deformations of a fiber correspond to complex deformations of a fiber in the mirror family.
We have to view the 1-parameter degenerations both algebraically and analytically. For example, the *Tate curve* \( \text{Tate} \) is a projective algebraic curve

\[
\{ y^2 + xy = x^3 + a_4(q)x + a_6(q) \} \subset \mathbb{P}^2(\mathbb{Z}[q])
\]

with a volume form \( \Omega = dx/(2y + x) \).

The series \( a_4, a_6 \in \mathbb{Z}[q] \) have radius of convergence 1, so, by viewing them as holomorphic functions in a variable \( q \) in the unit disc \( \Delta \subset \mathbb{C} \), we can make an analytic subvariety

\[
\text{Tate}^{an} \subset \mathbb{C}\mathbb{P}^2 \times \Delta.
\]

The fibers \( T_q \) of \( \text{Tate}^{an} \to \Delta \) for \( q \neq 0 \) are smooth of genus 1.

\( T_q \) carries a Kähler form obtained by pulling back the Fubini–Study form on \( \mathbb{C}\mathbb{P}^2 \), so it’s a symplectic manifold.

\( \text{Tate} \) is mirror to itself.
Projective 1-parameter CY degenerations

• A formulation of mirror pairs could identify a class of proper schemes

\[ X \to \text{Spec } \mathbb{Q}[[q]], \]

smooth after inverting \( q \), equipped with ample line bundles \( \mathcal{L} \) and trivializations \( \Omega \) of the dualizing sheaf; and an involution

\[ (X \to \text{Spec } \mathbb{Q}[[q]], \mathcal{L}, \Omega) \leftrightarrow (\check{X} \to \text{Spec } \mathbb{Q}[[q]], \check{\mathcal{L}}, \check{\Omega}). \]

• \( X \) and \( \check{X} \) should be defined as projective schemes over the ring of complex power series with positive radius of convergence, so that one can make analytic models

\[ X \subset \mathbb{C}P^N \times \Delta(r), \quad \check{X} \subset \mathbb{C}P^N \times \Delta(r). \]

• Gross–Siebert’s *toric degenerations program* provides algebraic mirror pairs, but convergence of the defining series is currently missing.

T. Perutz From seen to unseen Lagrangians via algebraic geometry
Here are names of some formulations of CY mirror symmetry.

\[ \text{toric-degenerate} \rightarrow \text{homological} \rightarrow \text{Hodge-theoretic} \rightarrow \text{enumerative} \]

- ✓ New!
- ✓ Easy
- Frobenius-algebraic

The arrows are implications, conjectured or proven.
Today, the standing assumption will be a *partial* statement of homological mirror symmetry (HMS)—it will not involve arbitrary coherent complexes, but only line bundles.

I’ll report that ‘partial HMS’ implies

1. *full* HMS;

2. *Frobenius-algebraic* mirror symmetry: an isomorphism of Frobenius algebras between quantum cohomology and tangential cohomology of the mirror; and therefore

3. *enumerative* mirror symmetry: a generating series that counts rational curves equals a Yukawa coupling on the mirror.
Homological mirror symmetry (HMS)

- HMS involves a version of the Fukaya $A_\infty$-category $\mathcal{F} = \mathcal{F}(X)$ of Lagrangian submanifolds $L \subset X$.
- I’ll be vague about the version I have in mind, because the results rely only on general properties. Among them, $\mathcal{F}$ should be a split-closed triangulated $A_\infty$ category, defined over a field $\mathbb{K}$ which contains $\mathbb{Q}[[q]][q^{-1}]$, with a weak CY structure $\mathcal{F} \simeq \mathcal{F}^\vee[-n]$.
- On the other side of the mirror we have a CY variety $\check{X}_\mathbb{K} \to \text{Spec } \mathbb{K}$, and (a DG model for) its derived category $D\check{X}_\mathbb{K}$ of bounded complexes of coherent sheaves.
- **HMS** asks for an $A_\infty$ quasi-equivalence $\mathcal{F} \simeq D\check{X}_\mathbb{K}$.
- **Partial HMS** involves the full subcategory $\mathcal{B} \subset D\check{X}_\mathbb{K}$ with objects $\{\check{L} \otimes r\}_{r \geq 0}$. It asks for a fully faithful $A_\infty$ embedding $\mathcal{B} \hookrightarrow \mathcal{F}$.
- More generally, one may take $\mathcal{B}$ to be any split-generating full subcategory of $D\check{X}_\mathbb{K}$.
Hypothesis: partial HMS with maximally unipotent monodromy

We suppose:
- Partial HMS holds: we have a fully faithful functor $\mathcal{B} \to \mathcal{F}$, where $\mathcal{B} \subset D(\check{X})$ is formed from powers of the polarizing line bundle.
- We have an analytic model
  \[
  \check{X} \subset \mathbb{C}P^N \times \Delta^*(r)
  \]
  for the mirror family. Write $\check{X}_q$ for the fiber over $q \in \Delta^*(r)$.
- The monodromy $T \in \text{Aut} H^n(\check{X}_q; \mathbb{C})$ (where $q \neq 0$) is \textit{maximally unipotent}, i.e.
  \[
  (T - I)^{n+1} = 0, \quad (T - I)^n \neq 0 \quad (n = \dim_{\mathbb{C}} \check{X}_q).
  \]

Up to a base-change $q \mapsto q^k$, it’s automatic that $(T - 1)^{n+1} = 0$. Frobenius-algebraic (or Hodge-theoretic) mirror symmetry implies $(T - 1)^n \neq 0$, so in that sense, our monodromy assumption does not reduce generality.
Theorem (P.–Sheridan.)

Assuming partial HMS with maximally unipotent monodromy, any chosen embedding $\mathcal{B} \to \mathcal{F}$ extends—uniquely, up to natural quasi-isomorphism—to a $\mathbb{K}$-linear quasi-equivalence $D\check{X}_K \simeq \mathcal{F}$.

- So to prove HMS, it’s enough to prove it for some collection of Lagrangians which in practice you can ‘see’—the mirrors to the line bundles $\check{\mathcal{L}} \otimes r$.
- $X$ might be teeming with other Lagrangians $L \subset X$ that you don’t see. But algebraically, they are built from the seen ones.
- The proof is short but invokes powerful tools:
  1. Abouzaid’s generation criterion for Fukaya categories;
  2. the Hochschild structure of algebraic varieties;
  3. the limiting mixed Hodge structure on the cohomology of a degenerating complex algebraic variety.
Theorem (P.–Sheridan.)

Assuming partial HMS with maximally unipotent monodromy, one has a canonical isomorphism of graded unital $\mathbb{K}$-algebras

$$\kappa: QH^*(X) \to HT^*(\check{X}_\mathbb{K})$$

from (small) quantum cohomology $QH^*(X) = H^*(X; \mathbb{K})$ to the tangential cohomology

$$HT^*(\check{X}_\mathbb{K}) = \bigoplus_{p+q=\ast} H^p(\Lambda^q T\check{X}_\mathbb{K})$$

which maps the symplectic class $[\omega] \in QH^2(X)$ to the Kodaira–Spencer class $\theta = KS(q(d/dq)) \in H^1(\mathcal{J})$.

Meaning of $\theta$: The family $\check{X} \to \Delta^*$ is a map $\gamma: \Delta^* \to \mathcal{M}$ into CY moduli space satisfying the ODE $q(d\gamma/dq) = \theta \circ \gamma$. 
Theorem (P.–Sheridan)

We have already stated that partial HMS with maximally unipotent monodromy implies that one has an isomorphism of graded unital $\mathbb{K}$-algebras

$$\kappa: QH^*(X) \to HT^*(\tilde{X}_K), \quad \kappa[\omega] = \theta.$$ 

In addition, $\kappa$ is a map of Frobenius algebras: for $c \in QH^{2n}(X)$, one has

$$\int_X c = \int_{\tilde{X}_q} \tilde{\Omega}_q \wedge (\kappa(c) \cdot \tilde{\Omega}_q) \in \mathbb{K}.$$ 

$\tilde{\Omega}_q$ is the restriction to $\tilde{X}_q$ of the unique relative volume $\tilde{\Omega}$ on $\tilde{X}_K \to \text{Spec} \mathbb{K}$ for which Floer–Poincaré duality corresponds under HMS to Serre duality:

$$HF(L_0, L_1) \cong HF(L_1, L_0)^\vee \iff \text{Ext} (\mathcal{E}_0, \mathcal{E}_1) \cong \text{Ext} (\mathcal{E}_1, \mathcal{E}_0)^\vee.$$
The following enumerative mirror symmetry statement follows from Frobenius-algebraic mirror symmetry ($\check{\Omega}$ as before):

**Corollary**

*Partial HMS with maximally unipotent monodromy implies that*

$$\int_X [\omega]^n = \int_{\check{\mathcal{X}}_q} \check{\Omega}_q \wedge \left( q \frac{d}{dq} \right)^n \check{\Omega}_q \in \mathbb{Q}[[q]].$$

We also have an exchange of summed Hodge numbers:

**Corollary**

*Partial HMS with maximally unipotent monodromy implies that*

$$b_k(X) = \sum_{i+j=k} \dim H^i(\Lambda^j \check{\mathcal{X}}_{\mathbb{K}}) = \sum_{i+j=k} h_{n-i,j}(\check{\mathcal{X}}).$$
Where are we?

- We’re missing a proof that HMS implies Hodge-theoretic mirror symmetry: an isomorphism of $\mathbb{Q}H^*(X)$ with the algebraic de Rham cohomology of $H^*_{DR}(\mathcal{X}_K)$ which respects variations of Hodge structure.
- We’re missing a proof that toric-degenerate mirror pairs obey partial HMS. However, Gross and Siebert have already presented the glimmerings of an argument.