Floer invariants of Anosov flows on 3-manifolds

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Joint with Kai Cieliebak, Agustin Moreno, Thomas Massoni

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Liouville manifolds

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- X has a contact structure at infinity $\partial_\infty X$ given by $\ker(\lambda|_{\partial_\infty X})$

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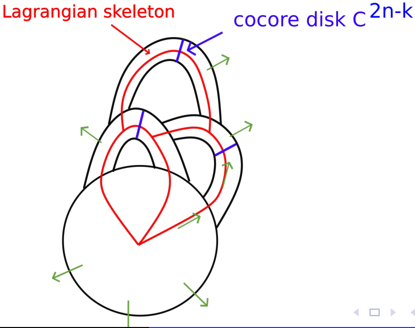
- Core of H^k is isotropic disks D^k; co-cores is co-isotropic disk D^{2n-k}; skeleton is union of cores = singular Lagrangian



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- For maximal index n handles, cocores are Lagrangian disks with Legendrian boundary
- Cocores transversely intersect skeleton at critical points of f = fixed points of Liouville vector field ∇f on skeleton

Wrapped Fukaya category of Weinstein manifolds

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  objects are (twisted complexes of) embedded exact Lagrangians $L \subset X$, closed or with Legendrian boundary $\partial L \subset \partial X$.
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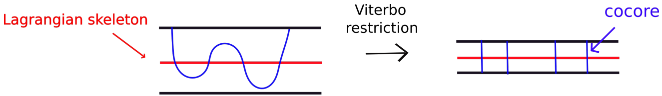
- **Proof idea:**



Anosov flows

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$$TM \cong < X > \oplus E^s \oplus E^u$$
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- F^{ws}, F^{wu} are *taut* foliations, so by Novikov's theorem, any tranverse loop is *non-contractible*

Bicontact structures from Anosov flows

- $\alpha_+ := \alpha_u + \alpha_s$ and $\alpha_- := \alpha_u - \alpha_s$ define two contact structures $\xi_+ := \ker \alpha_+$ and $\xi_- := \ker \alpha_-$ on M

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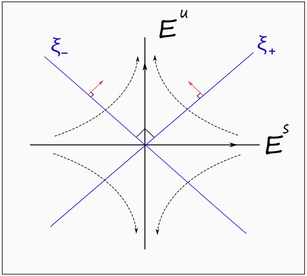

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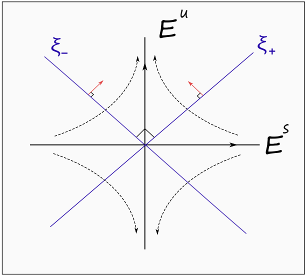

- ξ_+, ξ_- are *hypertight* contact structure: Reeb vector fields R_+, R_- of ξ_+, ξ_- respectively are positively transverse to F^{wu}, so any Reeb orbit is non-contractible by tautness of F^{wu}

Liouville manifolds from Anosov flows

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- $M \times \mathbb{R}_s$ is Liouville but not Weinstein: $H^3(M \times \mathbb{R}) \neq 0$, while 4-dimensional Weinstein manifolds have singular cohomology in degrees at most 2, the half-dimension

Lagrangians in Anosov Liouville manifolds

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- Analogous to Weinstein case, where fixed points of Liouville flow on skeleton gave rise to Lagrangian co-cores, which generate: **Question:** Do Lagrangian cylinders L_γ generate $\mathcal{W}(M \times \mathbb{R})$?

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  - If M is unit cotangent bundle $ST^*\Sigma$ with Anosov flow = geodesic flow, then ξ_+, ξ_- are prequantization contact structure and standard contact structure with filling $T^*\Sigma$; studied by **McDuff 1991**, first example of Liouville but not Weinstein manifold
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- Also, McDuff examples have closed exact tori for each closed *embedded* geodesic in Σ (at most $3g - 3$ such geodesics). Torus bundles have no orientable closed exact Lagrangians.

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- Lagrangians that are strictly exact near the skeleton $M \times 0 \subset M \times \mathbb{R}$ are generated by L_γ, i.e. $\mathcal{W}_{cyl}(M \times \mathbb{R}) \cong \mathcal{W}_{strict}(M \times \mathbb{R})$, by Viterbo restriction functor.

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$$HH^{nc}_*(\mathcal{W}_{cyl}) \to SH^{*+2}_{nc}(M \times \mathbb{R}) \tag{1}$$
$$HH^{c}_*(\mathcal{W}_{cyl}) \to SH^{*+2}_{c}(M \times \mathbb{R}) \tag{2}$$

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$$HH_*^{nc}(\mathcal{W}_{cyl}) \to SH_{nc}^{*+2}(M \times \mathbb{R}) \tag{1}$$

$$HH_*^{c}(\mathcal{W}_{cyl}) \to SH_{c}^{*+2}(M \times \mathbb{R}) \tag{2}$$

- The unit of SH^* is contained in SH_c^0
- On other hand, $HH_*^c(\mathcal{W}_{cyl}) \cong \oplus_\gamma HH_*(C^*(S^1 \times \mathbb{R}))$ is supported in degree 0, 1 by explicit computation; so the image of HH_*^c is in degrees 2, 3 and cannot contain the unit.

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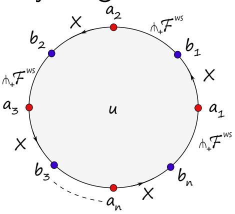
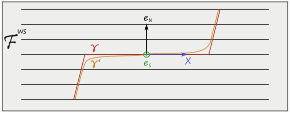
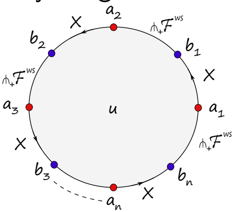
- To prove splittings, need to show that any J-holomorphic disk with inputs at least one chord, has output given by a chord
- Suppose there is a disk in $M \times \mathbb{R}$ whose boundary maps alternatively to L_γ and input Reeb chords; its projection to M has boundary tangent to X or Reeb chords.

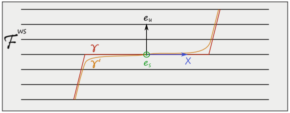




Topological disk lemma

- To prove splittings, need to show that any J-holomorphic disk with inputs at least one chord, has output given by a chord
- Suppose there is a disk in $M \times \mathbb{R}$ whose boundary maps alternatively to L_γ and input Reeb chords; its projection to M has boundary tangent to X or Reeb chords.





- **Topological disk lemma:** such a disk cannot exist.
- Reeb chords positively transverse to F^{ws}; also X is tangent to F^{ws} but can be perturbed to be positively transverse to F^{ws}, which contradicts the tautness of F^{ws}.

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$$HH_*^c(A) \to HH_*^c(\mathcal{W}_{cyl}) \tag{3}$$
$$HH_*^{nc}(A) \to HH_*^{nc}(\mathcal{W}_{cyl}) \tag{4}$$

- On the other hand,

$$HH_*^c(\mathcal{W}_{cyl}) \cong HH_*^c(A) \oplus \left(\oplus_{\gamma \notin A} HH_*(C^*(S^1))\right)$$

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- Since $HH_*(C^*(S^1))$ is non-zero, the map $HH_*(A) \to HH_*(\mathcal{W}_{cyl})$ cannot be isomorphism, and hence A cannot split-generated $\mathcal{W}_{cyl}$

Thanks you!

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