Flexibility and rigidity in contact and symplectic geometry

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The Haitian Scientific Society

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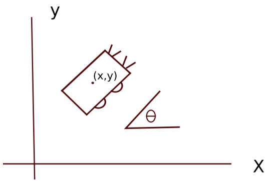

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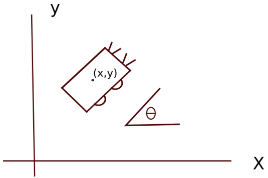

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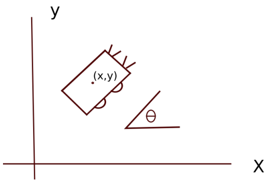

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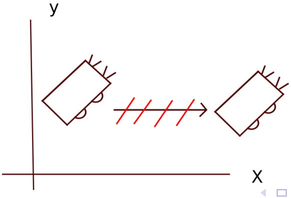

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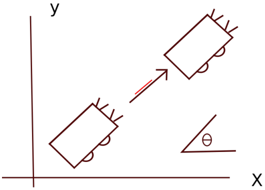

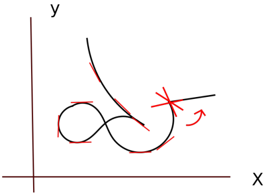

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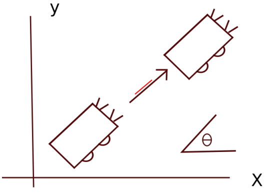

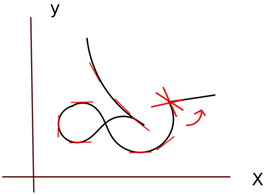

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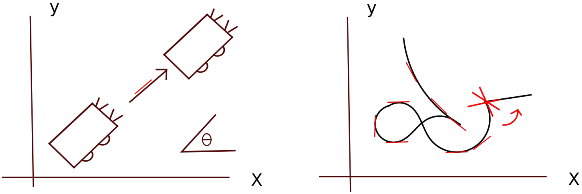

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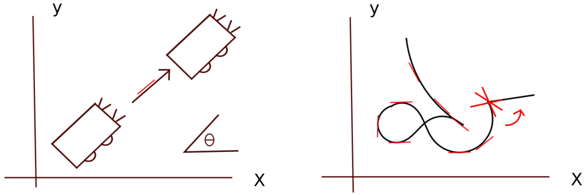

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- **Question:** can any path in $\mathbb{R}^3$ be approximated by the motion of a non-slipping car?

Formal/genuine functions

- Graph of function $z(x)$ with its derivative: $(x, \frac{dz}{dx}, z(x)) \subset \mathbb{R}^3_{x,p,z}$

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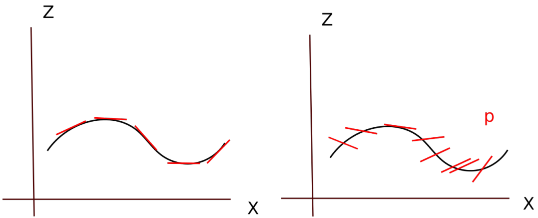

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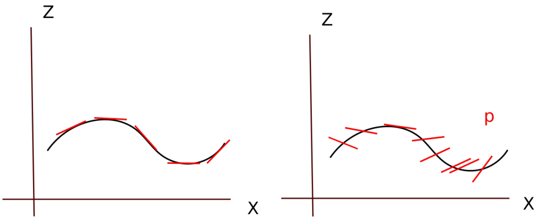

- $(x, y(x), z(x)) \subset \mathbb{R}^3$ is graph of 'genuine' function if $\frac{dz}{dx} = p$, i.e. tangent to the hyperplane distribution $\xi^2 := \ker(dz - pdx)$

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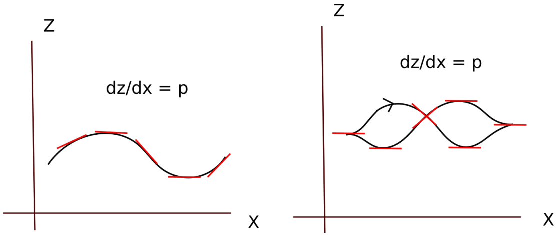

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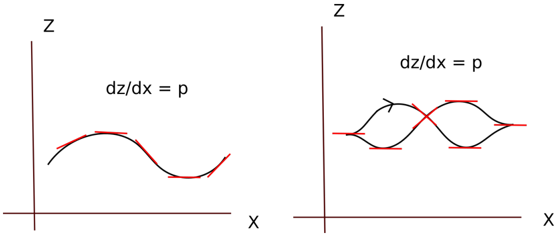

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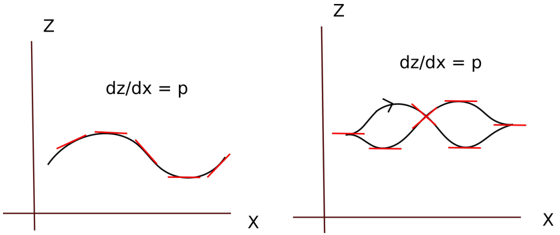

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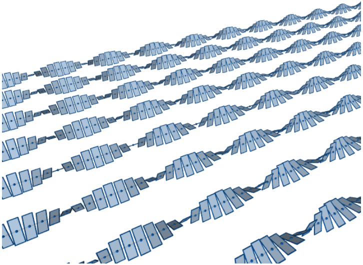

Figure: Contact distribution $\xi_{std} = \ker(dz - ydx) \subset T\mathbb{R}^3$, image by P. Massot.

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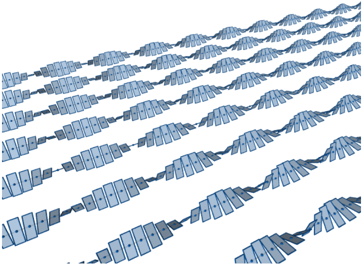

Figure: Contact distribution $\xi_{std} = \ker(dz - ydx) \subset T\mathbb{R}^3$, image by P. Massot.

- Observe that the contact planes ξ are very twisted (maximally non-integrable). Largest subspace that is tangent to ξ is 1-dimensional!

Contact geometry

- **Definition:** a *contact structure* ξ on a manifold Y^{2n+1} is a 2n-plane distribution $\xi^{2n} = \ker(\alpha)$ for a 1-form α with $\alpha \wedge (d\alpha)^n \neq 0$, *maximally non-integrable*

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- **Examples:** $(\mathbb{R}^{2n+1}, \xi)$, 1-jet space $J^1(M) = T^*M \times \mathbb{R}$

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- The (universal cover of the) previous two examples are contactomorphic: exists a map $\phi : (M, \xi_M) \to (N, \xi_N)$ taking ξ_M to ξ_N

Isotropics

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- Finding isotropics is equivalent to solving a PDE given by α, not easy! Ex. $\frac{dy}{dx} = \tan(\theta)$, or $\frac{dz}{dx} = y$

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- **Weinstein neighborhood theorem:** any Legendrian $\Lambda^n \subset (Y^{2n+1}, \xi)$ has a neighborhood that is equivalent to neighborhood of Λ in 1-jet space $J^1(\Lambda)$

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- h-principle is an example of flexibility

Partial Differential Relations in contact topology

- **Definition:** A *formal* contact structure is a 1-form α and a 2-form ω so that $\alpha \wedge \omega^n \neq 0$ (i.e. a non-degenerate 2-form ω on $\ker \alpha$). However, do *not* require $\omega \neq d\alpha$

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- Can a formal isotropic be deformed to a genuine isotropic?

Flexibility for isotropics

- **Gromov 1970's:** h-principle for subcritical isotropics: two formally isotopic $\Lambda^k_1, \Lambda^k_2 \subset (Y^{2n+1}, \xi)$ with $k < n$ are genuinely isotopic

Flexibility for isotropics

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- **Definition:** a Legendrian $\Lambda^n \subset Y^{2n+1}$ is *loose* if $n \geq 2$ and it has a 'zig-zag' in its *xz*-projection

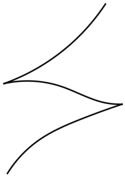

Figure: Loose chart, i.e. zig-zag, pictured in $\mathbb{R}^2_{xz}$

Loose Legendrians

- **Murphy's h-principle for loose Legendrians 2012:** formally isotopic loose Legendrians (in dimensions $n \geq 2$) are Legendrian isotopic; any smooth embedding can be C^0-approximated by a loose Legendrian.

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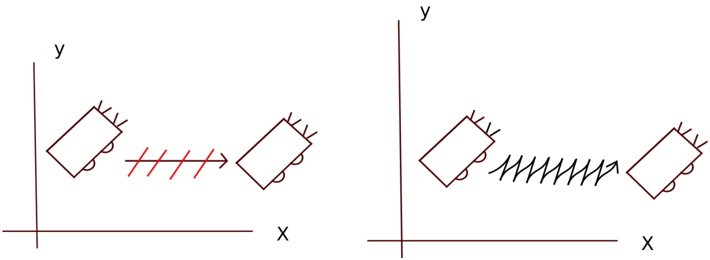

Figure: Approximating slipping path $(t, 0, \pi/4)$ by non-slipping path

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- Many Legendrian knots in $(\mathbb{R}^3, \xi_{std})$ are formally isotopic but not Legendrian isotopic, distinguished by LCH; h-principle fails

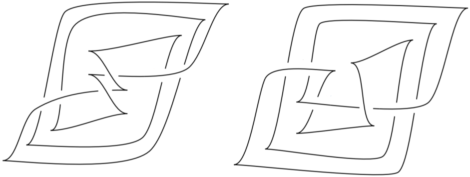

Figure: Chekanov Legendrians in $\mathbb{R}^2_{xz}$; images due to John Etnyre

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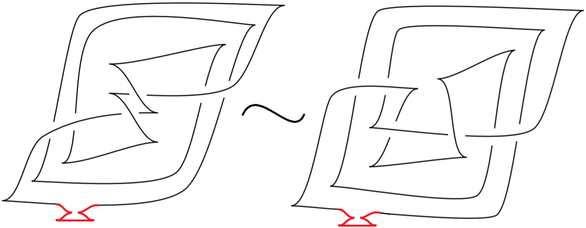

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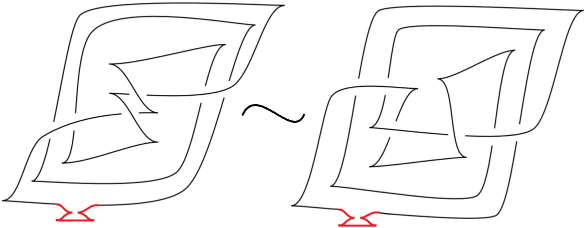

- **Open problem:** If Λ has vanishing LCH, is it loose?

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- **Theorem** (L., with Sylvan and Tanaka) for any Legendrian Λ in $(Y^{2n+1}, \xi), n \geq 3$, and any integer P, there is a 'P-loose' Legendrian Λ_P formally isotopic to Λ with $(\Lambda_P)_P \cong \Lambda_P$ and $LCH(\Lambda_P) \cong LCH(\Lambda)[\frac{1}{P}]$

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Interpolating between flexibility and rigidity for Legendrians

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- Motivated by construction in classical topology called rational homotopy theory.

Rigidity in contact geometry, II

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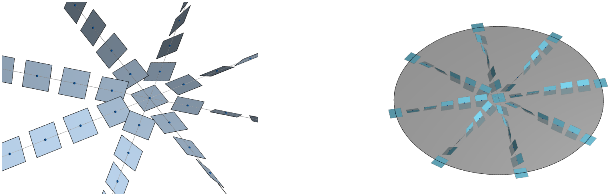

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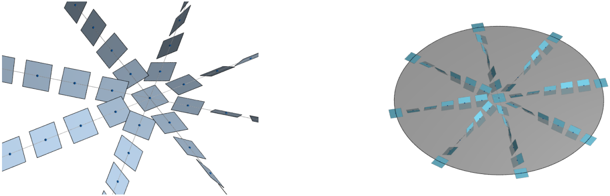

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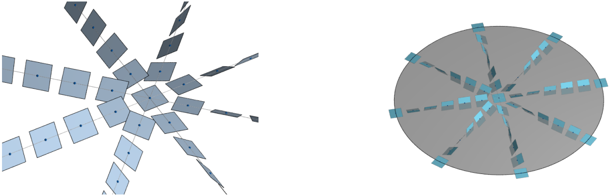

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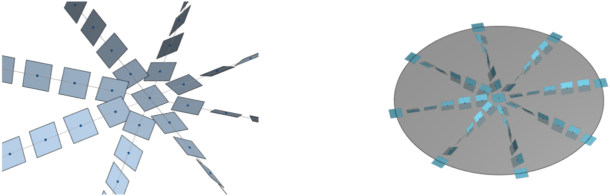

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- **Symplectic Darboux theorem:** any symplectic manifold (M, ω) is locally $(\mathbb{R}^{2n}, \sum_{i=1}^n dq_i \wedge dp_i)$

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- As for contact structures, one can define isotropics, formal symplectic structure, discuss flexibility/rigidity...

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- J-holomorphic curve invariants like Floer theory can give non-trivial lower bounds on the number of closed Hamiltonian orbits

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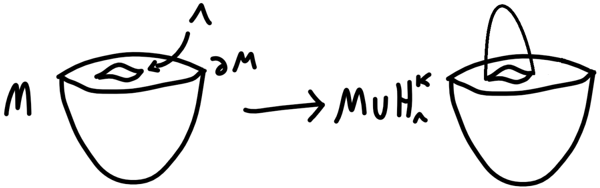

Figure: Weinstein handle attachment

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- **Example:** $T^*S^n = B^{2n} \cup H^n_{\Lambda_{unknot}}$
- **Theorem (Mclean)** There are infinitely many Weinstein structures on B^{2n}.

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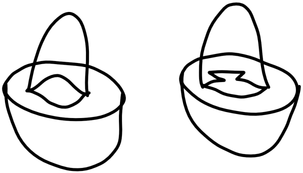

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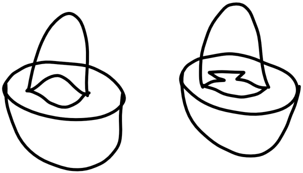

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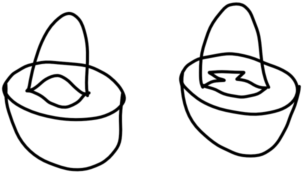

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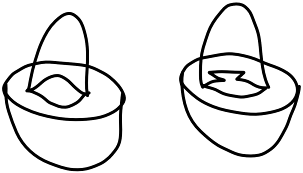

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- Use flexible techniques to create rigid contact structures.

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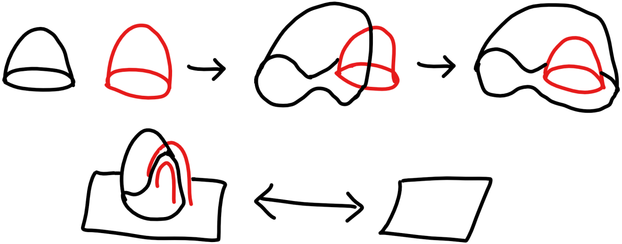

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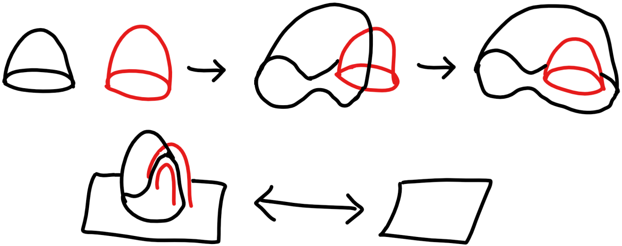

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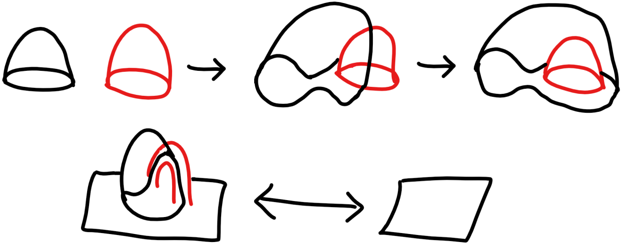

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Thank You!