A symplectic geometric view on syzygies

Oleg Lazarev
UMass Boston Mathematics Department

Syzygies and mirror symmetry seminar

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Cotangent bundles

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- v is a proper vector field, which implies that unit cotangent bundle ST^*M has a *contact* structure.
- Lagrangian $L \subset T^*M$ is exact if $\lambda|_L$ is an exact, i.e. $\lambda_L = dh$ for a function $h : L \to \mathbb{R}^1$. Legendrian boundary if $h|\partial L = 0$.
- Examples: take a smooth submanifold $H \subset M$. Then the conormal $N_H = \{\alpha \in T^*M | \alpha(TH) = 0\}$ is an exact Lagrangian with Legendrian boundary

Fukaya category

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- Consider exact Lagrangians $L \subset T^*M$ with Legendrian ∂L disjoint from Λ, and isotopies through such Lagrangians

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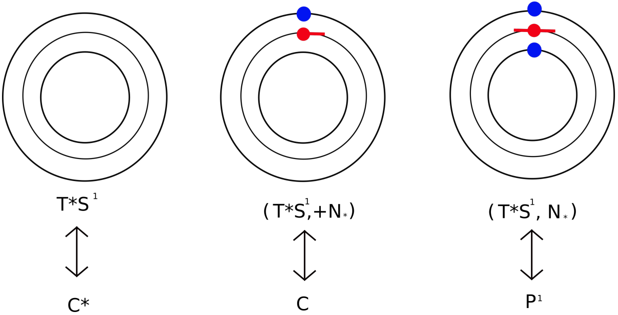
- Morphisms are Floer cochains generated by intersection points (need to consider all intersection points after "wrapping," applying a positive isotopies, disjoint from Λ)
- Make $W(T^*M, \Lambda)$ be (pre)-triangulated by taking (pre)-triangulated closure, i.e. forming twisted complexes $TwW(T^*M, \Lambda)$

Mirror symmetry

- Given a toric Y, there is a stopped cotangent bundle of torus $T^*(T^n, \Lambda)$ so that

$$Tw^{\pi} W(T^* T^n, \Lambda) \cong D^b Coh(Y)$$

- Λ will be a subset of the conormal of a stratification of T^n.



Lagrangian intersections with the zero-section

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For general, non-transverse intersections

$$|M \cap \phi(M)| \geq cuplength H^*(M; \mathbb{R}) + 1$$

Cuplength of a graded ring R: the largest n so that there exist positive degree elements $r_1, \cdots, r_n$ whose product is non-zero.

Lagrangian intersections via Morse theory

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- **Example:** consider $f : M \to \mathbb{R}$. Then $df \subset T^*M$ is an exact Lagrangian isotopic to M (via $d(tf)$) Then

$$M \cap df \leftrightarrow \text{critical points of f}$$

  and intersection is transverse iff f is a Morse function.

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- General case uses Floer theory, i.e. infinite-dimensional Morse theory

Viterbo restriction functor

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- A *Weinstein* domain is a Liouville domain for which exists a Morse function $f : X \to \mathbb{R}$ so that v_λ is the gradient of f. Weinstein domains have "symplectic handlebody decompositions" into $T^*D^k \times T^*D^{n-k}$

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- **Abouzaid-Seidel, Ganatra-Pardon-Shende:** If $i : X_0 \subset X$ are Weinstein and $i(X \backslash X_0)$ is a Weinstein cobordism, there is a Viterbo restriction functor

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which is a localization functor and takes L to $L \cap X_0$ if $L \cap X_0$ is connected.

Resolutions from Viterbo restriction

- If $L \cap X_0 = L_1 \coprod \cdots \coprod L_k \subset X_0$, then $i^*(L)$ is a twisted complex on $L_1 \coprod \cdots \coprod L_k$, i.e. $i^*(L)$ has a resolution $(\oplus_i L_i, d)$.

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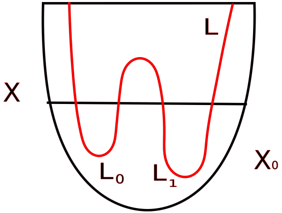

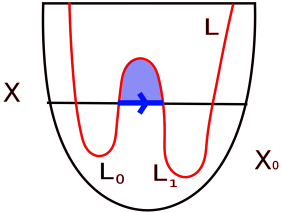

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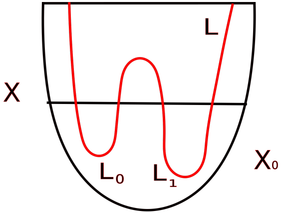

- **Example:** If $\Lambda_0 \subset \Lambda$, then $(T^*M, \Lambda_0) \subset (T^*M, \Lambda)$ is a subdomain. So there are localization functors

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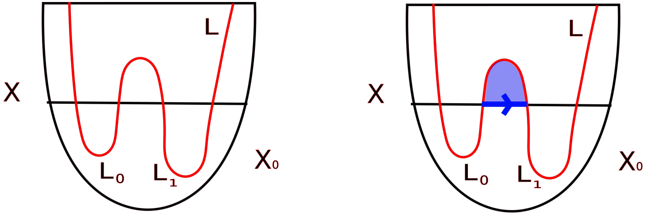

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which are mirror to the restriction localizations

$$D^b Coh(\mathbb{P}^1) \to D^b Coh(\mathbb{C}^1) \to D^b Coh(\mathbb{C}^*)$$

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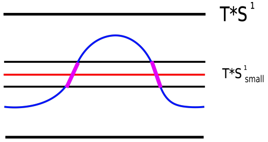
- Pick a Morse function $f : M \to \mathbb{R}$. Then $df \subset T^*M$ intersects M transversely and near M, df looks like a cotangent fibers; that is, $df \cap T^*M_{small}$ is a disjoint union of cotangent fibers.



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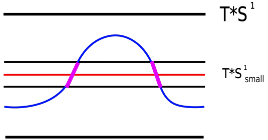

- **Corollary:** $M \cong df \cong i^*(df)$ has a resolution into cotangent fibers: $M \cong (\oplus_{x \in Crit(f)} T^*_x M, d)$. For example,

$$S^1 \cong Cone(T^*_x S^1 \overset{1-\gamma}{\to} T^*_y S^1) \leftrightarrow O_1 \cong Cone(O_{\mathbb{C}^*} \overset{1-x}{\to} O_{\mathbb{C}^*})$$

Rouquier dimension

- Let C be a (pre)-triangulated category and let G be a split-generator, i.e. all objects are iterated cones (and summands) of G.
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- **Rouquier:** $RDim(C) \leq DDim(C)$.
  Idea: given a resolution of $\Delta \cong (\oplus_i c_i \otimes c_i', d)$ into product objects, get a resolution of an arbitrary object c of C by convolution, i.e. $c \cong (\oplus_i Hom(c_i, c) \otimes c_i', d)$

Rouquier dimension in symplectic geometry

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  To bound $RDim(W(T^*M, \Lambda))$ need to consider Morse functions f so that df is disjoint from $\Lambda \times \mathbb{R}_{>0} \subset T^*M$.

Rouquier dimension for arbitrary Weinstein domains

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- Example: if $X = T^*M$, skeleton is the zero-section M.

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$$RDim(W(X)) \leq DDim(W(X)) \leq |\phi(c_X) \cap c_X| - 1$$

Lusternik-Schnirelmann category and Rouquier dimension

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- **Conjecture:** $RDim(W(T^*M)) = cuplength(H^*(M))$. **Hanlon-Hicks-L.** proved that minimum generation time using the fiber is at least cuplength, but need to do it for all generators.

Thank You!