Flexibility in Contact and Symplectic Geometry

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Michael Zhao Memorial Student Colloquium

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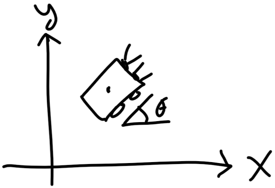

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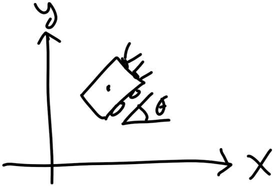

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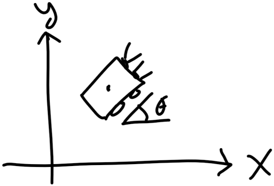

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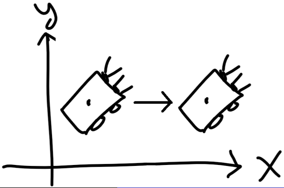

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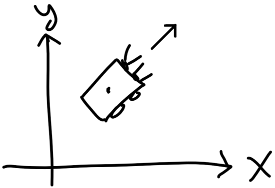

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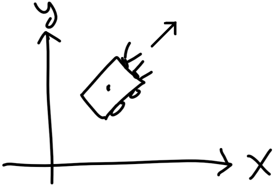

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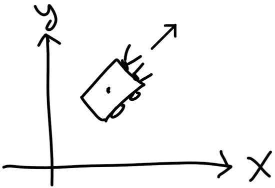

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- **Question:** can any path in $\mathbb{R}^3$ be C^0-approximated by the motion of a non-slipping car?

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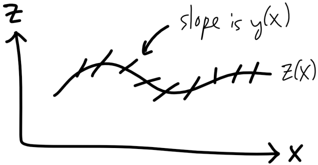

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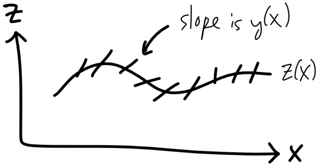

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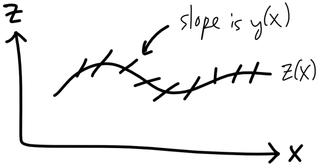

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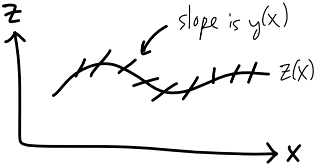

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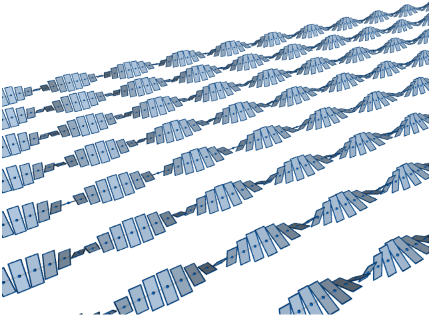

Figure: The contact distribution $\xi_{std} = \ker(dz - ydx) \subset T\mathbb{R}^3$, image due to Patrick Massot

Contact geometry

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- **Basic but important linear algebra fact**: if $\Lambda^k \subset (Y^{2n+1}, \xi)$ is isotropic, then $k \leq n$ (called Legendrian if $k = n$). Intuition: contact distribution is maximally non-integrable.

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- **Weinstein neighborhood theorem:** any Legendrian $\Lambda^n \subset (Y^{2n+1}, \xi)$ has a neighborhood that is contactomorphic to neighborhood of Λ in $J^1(\Lambda)$

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- **Question:** does h-principle hold for contact structures or isotropic submanifolds?

Rigidity in contact geometry

- There are non-local, deformation stable invariants of contact manifolds, Legendrians called *contact homology* and *Legendrian contact homology*, Gromov-Witten type invariant defined using J-holomorphic curves. Related to wrapped Fukaya category, mirror symmetry...

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- Many Legendrian knots in $(\mathbb{R}^3, \xi_{std})$ are formally isotopic but not Legendrian isotopic, distinguished by Legendrian contact homology

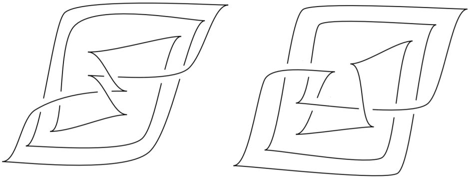
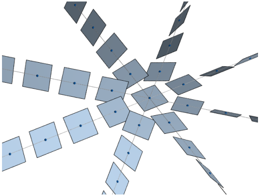


Figure: Chekanov Legendrians in $\mathbb{R}^2_{xz}$; images due to John Etnyre

Rigidity in contact geometry, II

- Similarly, many contact structures are formally contactomorphic but not contactomorphic



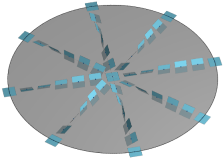


Figure: Standard and overtwisted structures; images due to Patrick Massot

- h-principle fails for contact manifolds, isotropic submanifolds! i_* is not injective on π_0; for Legendrian knots, i_* is not surjective on π_0
- **Question:** what is the boundary between rigidity and flexibility?

Flexibility for isotropics

- **Gromov's h-principle for subcritical isotropics:** two formally isotopic $\Lambda_1^k, \Lambda_2^k \subset (Y^{2n+1}, \xi)$ with $k < n$ are genuinely isotopic

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- h-principle fails for general Legendrians $(k = n)$ by LCH
- **Definition:** a Legendrian $\Lambda^n \subset Y^{2n+1}$ is *loose* if $n \geq 2$ and it has a 'zig-zag' in its *xz*-projection

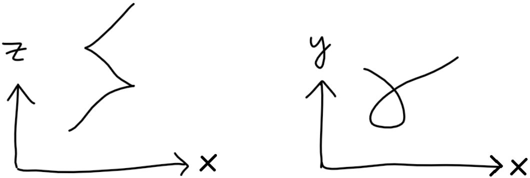

Figure: Loose chart, i.e. zig-zag, pictured in $\mathbb{R}^2_{xz}$ and in $\mathbb{R}^2_{xy}$

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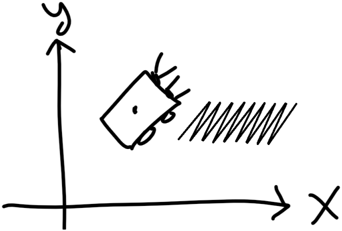

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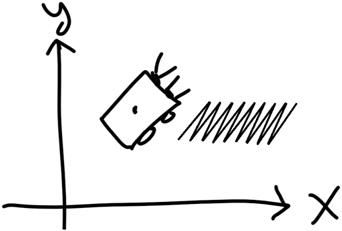

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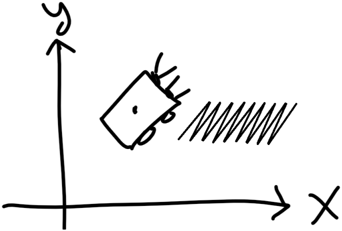

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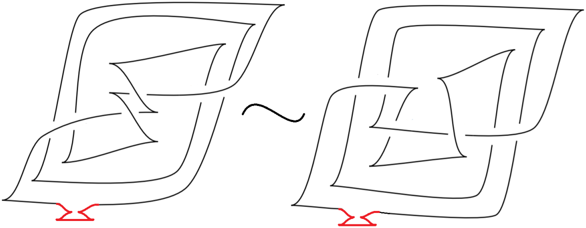
- **Open problem:** If Λ has vanishing LCH, is it loose?

Loose Legendrians, II

Loose Chekanov knots (in high-dimensions) are Legendrian isotopic



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- **Taubes:** i_* not surjective in dimension 4 by Seiberg-Witten = Gromov-Witten: $\mathbb{CP}^2 \sharp \mathbb{CP}^2 \sharp \mathbb{CP}^2$ has no symplectic structure

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Symplectic manifolds

- **Definition** a symplectic structure ω on a manifold M^{2n} is a closed, non-degenerate 2-form ω; get $[\omega] \in H^2(M; \mathbb{R})$ and $[\omega]^n \neq 0$ if M closed manifold
- **Examples:** Kahler manifolds, $(\mathbb{R}^{2n}, \sum_{i=1}^n dx_i \wedge dy_i)$, T^*M
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- **Open problems:** Is i_* injective on π_0 in dimension 4? Is i_* surjective on π_0 in dimensions > 4?

Weinstein domains

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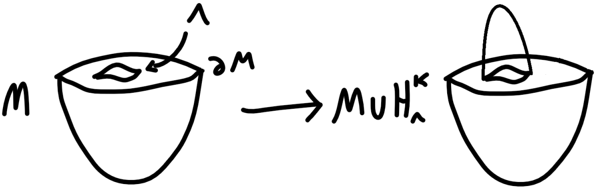

Figure: Weinstein handle attachment

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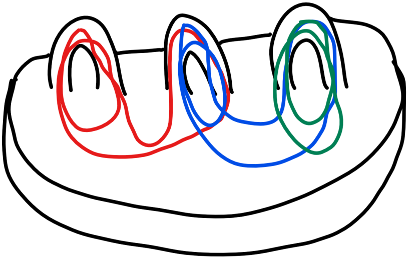

Figure: Sketch of an exotic Weinstein ball

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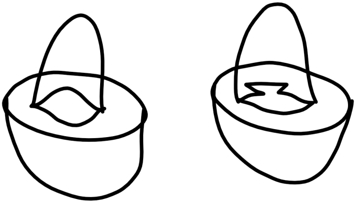

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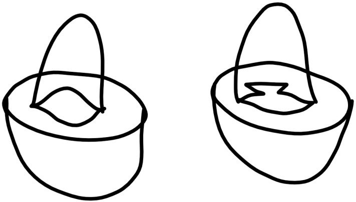

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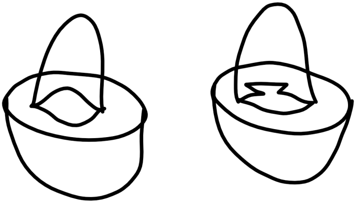

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- **Cieliebak-Eliashberg:** A formal Weinstein manifold W^{2n}, $n \geq 3$, has a genuine Weinstein structure. Two formally symplectomorphic flexible structures are symplectomorphic
- **Question:** can this result be used to construct symplectic structures on closed manifolds?

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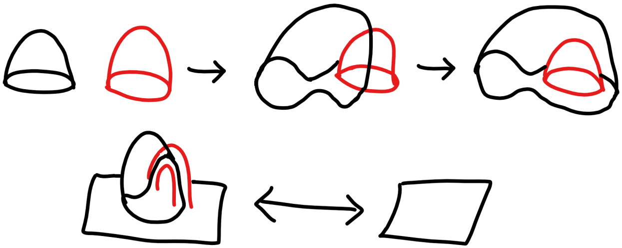

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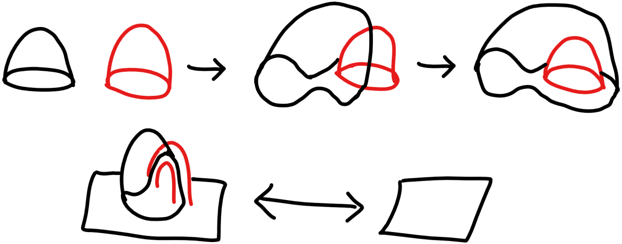

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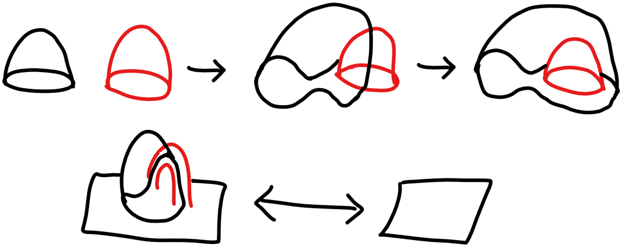

Figure: Handle-slides and handle cancellation/creation

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- **Question:** what is the interaction between symplectic flexibility and rigidity?