Quantum weak values are not unique
What do they actually measure?

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Abstract

Precise definitions of "weak [quantum] measurements" and "weak value" [of a quantum observable] are offered, which seem to capture the meaning of the often vague ways that these terms are used in the literature. Simple finite dimensional examples are given showing that weak values of an observable are not unique, and in fact arbitrary weak values can be obtained by appropriate weak measurements. This implies that a "weak value" of an observable A, *by itself*, can furnish no useful information about A; any information in a weak value is inextricably connected with the particular measurement procedure used to obtain the weak value.

A "Remarks" section questions the utility of "weak measurement".

1 Introduction and notation

We assume that the reader is familiar with the concept of "weak value" of a quantum observable. This concept was introduced in the seminal paper [3] of Aharonov, Albert, and Vaidman, called "AAV" below. It will be briefly reviewed below, and a much fuller presentation intended for those unfamiliar with weak values can be found in [1].

We attempt to stay as close as possible to traditional physics notation, reverting to notation more common in mathematics only when it seems less ambiguous or complicated. Our mathematical formulation of quantum mechanics generally follows that of Chapter 2 of the book of Nielsen and Chuang [2], with differences in notation noted below.

The inner product of vectors v, w in a complex Hilbert space H will be denoted $\langle v, w\rangle$, with the physics convention that this be linear in the second variable w, and conjugate-linear in the first variable v. The norm of a vector v will be denoted as $|v| := \langle v, v\rangle^{1/2}$.

Technically, a (pure) "state" of a quantum system with Hilbert space H is an equivalence class of nonzero vectors in H, where vectors $v, w \in H$ are equivalent if and only if $w = \alpha v$ for some nonzero constant α. However, we informally refer to vectors in H as "states", or "pure states", when we need to distinguish between pure states and "mixed states" (see below). A state v is said to be *normalized* if $|v| = 1$. We do not assume that states are necessarily normalized.

The projector to a subspace E will be denoted P_E, in place of the common but unnecessarily complicated physics notation $\sum_i |e_i\rangle\langle e_i|$, where $\{e_i\}$ is an orthonormal basis for E. When E is the entire Hilbert space of states, P_E

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is called the *identity operator* and denoted $I := P_E$. When E is the one-dimensional subspace spanned by a vector w, we may write P_w for P_E. When $|w| = 1$, $P_w v = \langle w, v\rangle w$, but the reader should keep in mind that under our convention, $P_w = P_{w/|w|}$, so this formula for P_w only applies for $|w| = 1$.

Mixed states are represented by "density matrices" $\rho : H \to H$, which are defined as positive operators on H of trace 1. A pure state $h \in H$ corresponds to the density matrix P_h.

We shall be dealing with a quantum system S in which we are primarily interested, which will be coupled to a quantum "meter system" M. We make no notational distinction between the physical systems S and M and their Hilbert spaces.

The composite system of S together with M is mathematically represented by the Hilbert space tensor product $S \otimes M$. We assume the reader is generally familiar with the mathematical definition of $S \otimes M$. The highlights of the definition are as follows.

Some, but not all, vectors in $S \otimes M$ can be written in the form $s \otimes m$ with $s \in S$ and $m \in M$; these are called "product states". Typical physics notation for $s \otimes m$ might be $|s\rangle|m\rangle$ or $|s\rangle_S|m\rangle_M$. Every vector v in $S \otimes M$ is a (possibly infinite) linear combination of product states : $v = \sum_i s_i \otimes m_i$.

If ρ is a density matrix on $S \otimes M$, its partial trace with respect to M, denoted $\mathrm{tr}_M\, \rho : S \to S$ is a density matrix on S. The (mixed) state of S corresponding to ρ is $\mathrm{tr}_M\, \rho$.

2 "Weak" measurements of a quantum observable

This section will review and formalize the notion of "weak measurement protocol". This is preliminary to the concept of "weak value" of a quantum observable, which is the main subject of this paper.

Let S denote the Hilbert space of a quantum system (also denoted S) and $A : S \to S$ a quantum observable (Hermitian operator) on S. To avoid technical mathematical issues, we assume throughout that S is finite dimensional, and until further notice, we shall also assume that the meter system M is finite dimensional.

A measurement of the observable A when the system S is in (pure) state $s \in S$ will change ("project") the pre-measurement state s to one of the eigenstates of A. Thus is is impossible in general to measure A without changing the state of S. However, AAV devised a clever way to measure the average value $\langle s, As\rangle$ of A in the normalized state s without significantly changing s. This will be called a "weak" measurement, which will be more precisely defined below. The next paragraphs describe a more general formulation of AAV's method.

Let M be the Hilbert space of a "meter" system which will be specially tailored below to measure $\langle s, As\rangle$. Then the Hilbert space of the composite system of S together with M is the tensor product $S \otimes M$.

The measurement of $\langle s, As\rangle$ will be accomplished by measurement in $S \otimes M$ of an observable of the form $I \otimes B$, where I denotes the identity operator on S, and $B : M \to M$ is a Hermitian operator on M. We shall refer $I \otimes B$ as the "meter observable" and we think of measuring $I \otimes B$ as "reading the meter".

Actually, we shall not be concerned with the detailed measurements of $I \otimes B$ (each of which would result in one of the eigenvalues of B), but rather the average value $\langle r, (I \otimes B)r\rangle$ of $I \otimes B$ in a slightly entangled state $r \in S \otimes M$.[2] The state r is obtained as follows.

Let $m \in M$ be a given "meter state". When M is in state m and S in state s, then the composite system $S \otimes M$ is in the product state $s \otimes m$. To avoid nuisance normalization factors, assume both s and m are normalized: $|s| = 1 = |m|$. It is traditional (though unnecessary) to apply a unitary operator to $s \otimes m$ to obtain the slightly entangled state r.[3] In the literature, the unitary operator is generally considered to be a time evolution operator written as e^{-iHt}, where H is the Hamiltonian. We shall consider the time t as a small positive parameter and emphasize this by writing ϵ instead of t. In this notation,

$$r = r(\epsilon) := e^{-i\epsilon H}(s \otimes m) \quad , \tag{1}$$

with H to be defined later.

Since $r(\epsilon) = e^{-i\epsilon H}(s \otimes m) = s \otimes m + O(\epsilon)$, and $\langle s \otimes m, (I \otimes B)(s \otimes m)\rangle = \langle s, Is\rangle\langle m, Bm\rangle = \langle m, Bm\rangle$, the average value $\langle r(\epsilon), (I \otimes B)r(\epsilon)\rangle$ of $I \otimes B$ in the state $r(\epsilon)$ will be of the form

$$\langle r(\epsilon), (I \otimes B)r(\epsilon)\rangle = \langle m, Bm\rangle + O(\epsilon) \quad . \tag{2}$$

We shall assume that

$$\langle m, Bm\rangle = 0 \quad . \tag{3}$$

Physically this corresponds to the assumption that the meter initially reads zero, on average. Under this assumption, $\langle r(\epsilon), (I \otimes B)r(\epsilon)\rangle = O(\epsilon)$, so we normalize the meter reading by dividing by ϵ:

$$\text{normalized average meter reading} := \frac{\langle r(\epsilon), (I \otimes B)r(\epsilon)\rangle}{\epsilon} \quad . \tag{4}$$

Eventually, we shall take a limit as ϵ goes to 0.

Our goal is to choose B such that

$$\lim_{\epsilon \to 0} \frac{\langle r(\epsilon), (I \otimes B)r(\epsilon)\rangle}{\epsilon} = \langle s, As\rangle \quad , \tag{5}$$

[2] Because of the slight entanglement, a measurement of $I \otimes B$ in state r will give some information about a measurement of $A \otimes I$ in state r. Since we are only speaking descriptively here, we do not need to defined "slightly entangled"; for the curious, its meaning is that the state is close to a product state without actually being a product state. (The definition of "entangled state" is one which is *not* a product state.)

[3] Some authors, such as AAV, use mathematical formulations which may look superficially different but are mathematically equivalent, as discussed briefly in a subsequent footnote and in detail in [1].

which says that the normalized average of the meter observable B approximates the average of A.

This is easily done by choosing H of the form

$$H := A \otimes G \quad \text{where } G : M \to M. \tag{6}$$

This makes

$$r(\epsilon) := e^{-i\epsilon H}(s \otimes m) = s \otimes m - i\epsilon As \otimes Gm + O(\epsilon^2), \tag{7}$$

so

$$\lim_{\epsilon \to 0} \frac{\langle r(\epsilon), (I \otimes B) r(\epsilon)\rangle}{\epsilon} = -i\langle m, (BG - GB)m\rangle\langle s, As\rangle, \tag{8}$$

and we need only choose m, B, and G such that

$$1 = -i\langle m, (BG - GB)m\rangle = 2\Im\langle m, BGm\rangle \quad , \tag{9}$$

where $\Im$ denotes imaginary part. Later we shall show how this can be done using a two dimensional meter space. Assuming that, the normalized average meter reading (4) will approximate $\langle s, As\rangle$ for small ϵ.

In addition, equation (7) suggests that the post-measurement state of S should differ only slightly from s for small ϵ. (This will be examined below; for the moment we assume it.) That will be what we mean by a "weak" measurement of $\langle s, As\rangle$. For future reference, we formalize the discussion with the following definition.

Definition 1 *Let S be a quantum system and A an observable on S. Let s be a state of S of which we can make an arbitrary number of copies. A* weak measurement protocol *is a procedure which can determine the expectation $\langle s, As\rangle$ of A in the state s to arbitrary accuracy while perturbing each copy of s used in the procedure by an arbitrarily small amount.*

Assuming that we can attain (9), and assuming the weakness of the measurement, the above discussion describes a "weak measurement protocol" in the sense of the definition.

"Weakness" of the measurement means that the state of S corresponding to the state of $S \otimes M$ after measurement of $I \otimes B$ is close to s for small ϵ.[4] To see this, first recall that measurement of $I \otimes B$ will change the state $r(\epsilon) := e^{-i\epsilon H}(s \otimes m)$ by projecting it onto one of the eigenspaces of $I \otimes B$. Each such eigenspace is the range of a projector $I \otimes P_Q$, where Q is an eigenspace of B. For small ϵ the state $r(\epsilon)$ is close to $s \otimes m$, so $(I \otimes P_Q)r(\epsilon)$ is close to $(I \otimes P_Q)(s \otimes m) = s \otimes P_Q m$. The state of S (expressed as a density matrix) is then close to $\mathrm{tr}_M\, P_{s \otimes P_Q m} = P_s$, which corresponds to the vector state s.

[4] Since everything in sight is finite dimensional and all norms are equivalent in finite dimensions, "close" may be interpreted relative to any convenient norm.

3 "Weak values" of a quantum observable

The last section defined "weak measurement protocol" and outlined an example of such, leaving details of its verification to later. Next we discuss how such weak measurement protocols are typically used in the literature to obtain "weak values" of a quantum observable.

Before starting, we warn the reader that we think that typical characterizations of "weak values" involve an implicit logical fallacy. To expose this fallacy, we shall need to present this characterization as it is typically done in the literature (though usually less explicitly), and therefore will sometimes need to temporarily use language which is logically questionable. When using such language, we shall point out that it is questionable.

We continue the discussion of the weak measurement protocol given in the previous section. We saw that for a system $S \otimes M$ in state $r(\epsilon) := e^{-i\epsilon H}(s \otimes m)$ (with H as specified there), the normalized average value of $I \otimes B$, $\langle r(\epsilon), (I \otimes B) r(\epsilon)\rangle/\epsilon$, approximates the average value $\langle s, As\rangle$ of A in state s for small ϵ. Moreover, for small ϵ, after the measurement of $I \otimes B$ in state $r(\epsilon)$, the state of S is close to its initial state s.

Suppose that immediately after measuring $I \otimes B$, we "postselect" to a given final state f. Mathematically, postselection corresponds to measuring $P_f \otimes I$ in $S \otimes M$. The physical picture is that we are asking the question "Is S in state f?". If the answer is "yes", we say that the postselection was successful and record the previous measurement of $I \otimes B$, if "no" (in which case S is in a mixture of pure states orthogonal to f), we discard the previous measurement.

Averaging the recorded measurements gives the conditional expectation of $I \otimes B$ in state $r(\epsilon)$ given successful postselection to f, which we shall denote by $E_\epsilon(B|f)$:

$$E_\epsilon(B|f) := \frac{\langle r(\epsilon), (P_f \otimes B) r(\epsilon)\rangle}{\langle r(\epsilon), (P_f \otimes I) r(\epsilon)\rangle} \quad . \tag{10}$$

Here we are modeling measuring B then immediately postselecting as measuring $P_f \otimes B = (P_f \otimes I)(I \otimes B)$. (We could also think of the measurement of B and postselection as occurring simultaneously; since $P_f \otimes I$ and $I \otimes B$ commute, we can measure them simultaneously.)

The conditional expectation $E_\epsilon(B|f)$ is $O(\epsilon)$. It is tempting to normalize by dividing by ϵ and take a limit as $\epsilon \to 0$, resulting in a normalized conditional expectation which we denote by $NE(B|f)$,

$$NE(B|f) := \lim_{\epsilon \to 0} \frac{E_\epsilon(B|f)}{\epsilon} \quad , \tag{11}$$

and then identify $NE(B|f)$ with the conditional expectation of A in state s given successful postselection. (We think this identification fallacious, for reasons given below.)

The normalized conditional expectation $NE(B|f)$ is called a "**weak value**" of A.[5] Note, however, that this terminology may be potentially misleading

[5]The term "weak value" is rarely precisely defined in the literature. Some authors introduce

because the "weak value of A" might depend not only on A (and other data associated with the original system S of interest such as $s, f \in S$), but also on data external to S such as the meter observable B and the meter state m.

Most of the traditional literature calculates this "weak value" as[6]

$$\Re \frac{\langle f, As \rangle}{\langle f, s \rangle} \quad , \tag{12}$$

where $\Re$ denotes real part. It is usually called something like "the weak value of A in initial state s and postselected state f," with all references to data external to S suppressed. This risks leaving the impression that "weak values" measure something intrinsic to S, independently of details of the meter system.

Using language which ignores the meter system may seem superficially reasonable because the normalized expectation of $I \otimes B$ in state $r(\epsilon)$ unquestionably approximates the expection $\langle s, As \rangle$ of A in state s, as shown in equation (5), independently of details of the meter system. This is as it should be for a good meter.

For example, all voltmeters which draw sufficiently small current (the analog of sufficiently weak coupling) are expected to read the about the same voltage. If meter A read "voltage" 50 and meter B read "voltage" 700 and both disturbed the system being measured negligibly (i.e., drew about the same small current), one would conclude that the two meters were measuring different things, of which at least one and possibly both should not be called "voltage".

Postselection replaces the (normalized) expectation (5) of the meter observable (which is also the expectation $\langle s, As \rangle$ of A) with the (normalized) conditional expectation $NE(B|f)$ (which is not obviously identifiable with anything intrinsic to the system S such as a conditional expectation of A). This makes it questionable to suppress details of the meter from the language.

We think that a transition from unconditional to conditional expectation of A would involve a logical fallacy which we now discuss in detail.[7]

Consider the two sentences:

1. The average value of the normalized (i.e., by division by ϵ) meter measurement equals the average value of A (in the limit $\epsilon \to 0$).

2. The average value of the normalized meter measurement conditioned on postselection to f equals the average value of A conditioned on postselection to f (in the limit $\epsilon \to 0$).

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it in ways which may seem different but are conceptually almost identical. A full discussion of how to recast the formulation of AAV into the terms above is given in [1].

[6]AAV [3] calulates it as $\langle f, As \rangle / \langle f, s \rangle$, a quantity which need not even be real, though a footnote in that paper suggests that the authors realize that the real part should be taken. Though subsequent literature sometimes quotes the AAV formula, the real part seems to be always taken in actually applying it. Ref. [1] explicitly identifies a possible error in the mathematics of AAV which could result in the omission of the real part.

[7]Whether the "weak value" literature actually assumes this fallacy may be controversial. The only paper known to me which clearly does not is [7]. Nearly all papers on "weak values" do seem to me to suggest some identification of the traditional weak value (12) with an intrinsic property of A (and f and s), though many are so vaguely written that it can be hard to say precisely what they might have intended.

Here "meter measurement" refers to measurement of $I \otimes B$.

We have seen that Sentence 1 is true. But Sentence 2 is either false, meaningless, or tautological, depending on how it is interpreted.

In order to speak meaningfully of "the average value of A conditioned on postselection to f", we need to say how this quantity is measured. If we measure A in S, successfully postselect to f, and average the results, we do *not* necessarily obtain the traditional "weak value (12), $\Re(\langle f, As\rangle / \langle f, s\rangle)$ (which is obtained from a specific choice of B and G). This is a simple calculation which will be done below.

Moreover, we shall show that many *different* "weak values" can be obtained with other choices of B and Hamiltonian $H := A \otimes G$ (i.e., weak values are not unique). This shows that independently of the correctness of calculations of the traditional "weak value" (12), sentence 2 cannot be true if "the average value of A conditioned on postselection to f" refers to normal measurements in S (without reference to the meter system M).

To what could it refer? If it refers to measurement in M of the normalized average value of B postselected to f (in the limit $\epsilon \to 0$), then sentence 2 becomes a tautology, true by definition and containing no useful information.

We now perform the simple calculation of the average value of A conditioned on postselection to f, assuming that we are referring to normal measurements in S (forgetting for the moment about the meter system M). For notational simplicity, we assume that A has just two distinct eigenvalues $\alpha_1 \neq \alpha_2$ with corresponding normalized eigenvectors a_1, a_2.

After A is measured, S is in state a_i with probability $|\langle a_i, s\rangle|^2$, $i = 1, 2$. Subsequently, the postselection succeeds with (conditional) probability $|\langle f, a_i\rangle|^2$. The total probability that the postselection succeeds is

$$\sum_{i=1}^{2} |\langle a_i, s\rangle|^2 |\langle f, a_i\rangle|^2 \quad .$$

Hence the conditional expectation of A given that the postselection succeeds is

$$\frac{\alpha_1 |\langle a_1, s\rangle|^2 |\langle f, a_1\rangle|^2 + \alpha_2 |\langle a_2, s\rangle|^2 |\langle f, a_2\rangle|^2}{|\langle a_1, s\rangle|^2 |\langle f, a_1\rangle|^2 + |\langle a_2, s\rangle|^2 |\langle f, a_2\rangle|^2} \quad . \tag{13}$$

It is easy to construct examples in which this does not equal the traditional "weak value" (12), $\Re(\langle f, As\rangle / \langle f, s\rangle$. For example, (13) is a convex linear combination of α_1, α_2, whereas it is well known that (12) need not be. This is emphasized by the provocative title of AAV [3]: "How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100". (When (12) is not a convex linear combination of the eigenvalues of A, it is called a "strange" weak value.)

4 Weak values are not unique

The last section exposed what we believe to be a logical fallacy implicit in the very concept of "weak value". Perhaps not all will accept this, but even those who do may wonder if the traditional "weak value" (12), $\Re(\langle f, As\rangle/\langle f, s\rangle)$, expresses some intrinsic property of the system S. The traditional literature would encourage such a belief because the only "weak values" which appear, to our knowledge, are (12) with or without the real part, and experiments verify only (12).

If other "weak values" can be obtained by similar reasoning, then that would cast doubt on any belief that (12) expresses some intrinsic property of observable A in system S. This section continues the reasoning of the "weak measurements" section to calculate the traditional "weak value" (12) and then calculates other "weak values" by similar reasoning. Notation is as defined there. Recall in particular that

$$r(\epsilon) := e^{-i\epsilon H(s\otimes m)} = e^{-iA\otimes G}(s\otimes m) = s\otimes m - i\epsilon As\otimes Gm + O(\epsilon^2) \qquad (14)$$

with $H := A\otimes G$ and both G and the meter observable B yet to be defined.

We shall calculate the normalized conditional expectation $NE(B|f)$ of the meter reading given postselection to f defined in equations (10) and (11): by:

$$NE(B|f) := \lim_{\epsilon\to 0}\frac{1}{\epsilon}\frac{\langle r(\epsilon), (P_f\otimes B)r(\epsilon)\rangle}{\langle r(\epsilon), (P_f\otimes I)r(\epsilon)\rangle}. \qquad (15)$$

First we calculate, recalling assumption (3) that $\langle m, Bm\rangle = 0$,

$$\begin{aligned}
&\frac{\langle r(\epsilon), (P_f\otimes B)r(\epsilon)\rangle}{\epsilon} \\
&= \frac{1}{\epsilon}\left[\langle s, P_f s\rangle\langle m, Bm\rangle + \langle s\otimes m, -i\epsilon P_f As\otimes BGm\rangle\right. \\
&\qquad \left. + \langle -i\epsilon As\otimes Gm, P_f s\otimes Bm\rangle + O(\epsilon^2)\right] \\
&= \langle s, -iP_f As\rangle\langle m, BGm\rangle + \langle -iAs, P_f s\rangle\langle Gm, Bm\rangle + O(\epsilon) \\
&= -i[\langle f, As\rangle\langle s, f\rangle\langle m, BGm\rangle - \langle f, s\rangle\langle As, f\rangle\langle Gm, Bm\rangle] + O(\epsilon) \\
&= -i[\langle f, As\rangle\langle s, f\rangle\langle m, BGm\rangle - \langle As, f\rangle\langle f, s\rangle\langle BGm, m\rangle] + O(\epsilon). \qquad (16)
\end{aligned}$$

Next note that

$$\lim_{\epsilon\to 0}\langle r(\epsilon), (P_f\otimes I)r(\epsilon)\rangle = \langle s\otimes m, (P_f\otimes I)(s\otimes m)\rangle = \langle s, P_f s\rangle = \langle f, s\rangle\langle s, f\rangle.$$

Combining this with (16) gives

$$\begin{aligned}
NE(B|f) &= \frac{-i[\langle f, As\rangle\langle s, f\rangle\langle m, BGm\rangle - \langle As, f\rangle\langle f, s\rangle\langle BGm, m\rangle]}{\langle f, s\rangle\langle s, f\rangle} \\
&= -i\left[\frac{\langle f, As\rangle}{\langle f, s\rangle}\langle m, BGm\rangle - \frac{\langle As, f\rangle}{\langle s, f\rangle}\langle BGm, m\rangle\right] \\
&= 2\Im\frac{\langle f, As\rangle\langle m, BGm\rangle}{\langle f, s\rangle} \quad , \qquad (17)
\end{aligned}$$

when $\langle f, s\rangle \neq 0$ and is undefined if $\langle f, s\rangle = 0$. The following calculations assume $\langle f, s\rangle \neq 0$.

We still need to specify M and m and B and G. We have been operating under assumption (3) that $\langle m, Bm\rangle = 0$. To this we must adjoin condition (9), namely

$$\Im\langle m, BGm\rangle = \frac{1}{2} \quad , \tag{18}$$

to assure that the normalized average measurement of $I \otimes B$ be $\langle s, As\rangle$. Write

$$\langle m, BGm\rangle = \rho + \frac{1}{2}i \quad \text{with } \rho \text{ real.} \tag{19}$$

Then (17) becomes

$$NE(B|f) = 2\Im\frac{\langle f, As\rangle\langle m, BGm\rangle}{\langle f, s\rangle} = \Re\frac{\langle f, As\rangle}{\langle f, s\rangle} + 2\rho\Im\frac{\langle f, As\rangle}{\langle f, s\rangle} \quad . \tag{20}$$

The first term in (20) is the "usual" weak value $\Re(\langle f, As\rangle/\langle f, s\rangle)$, but we shall show that when $\Im(\langle f, As\rangle/\langle f, s\rangle) \neq 0$, the second term can be chosen arbitrarily by adjusting the value of $\rho = \Re\langle m, BGm\rangle$.

To see that any number can be obtained for $\Re\langle m, BGm\rangle$ with $\langle m, Bm\rangle = 0$ and $\Im\langle m, BGm\rangle = 1/2$, take the meter space M to be two-dimensional with orthonormal basis $m, m^\perp$, and define G and B by the following matrices with respect to this basis:

$$G := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & \rho + i/2 \\ \rho - i/2 & 0 \end{bmatrix} \tag{21}$$

with ρ real. Then BG has the following form, where entries denoted "$*$" have not been calculated because they are irrelevant to calculation of $\langle m, BGm\rangle$ (which is the upper left entry of BG):

$$BG = \begin{bmatrix} \rho + i/2 & * \\ * & * \end{bmatrix} = \begin{bmatrix} \langle m, BGm\rangle & * \\ * & * \end{bmatrix}. \tag{22}$$

This shows that when $\Im(\langle f, As\rangle/\langle f, s\rangle) \neq 0$, by varying ρ, one can obtain any number whatever as a "weak value" for A: weak values are not unique.

If it happens that $\Im(\langle f, As\rangle/\langle f, s\rangle) = 0$, then the traditional "weak value" $\Re(\langle f, As\rangle/\langle f, s\rangle)$ is uniquely obtained *by the above method.* However, other methods can yield other "weak values". An example is given in [1].

5 Obtaining non-traditional "weak values" in the context of AAV

Von Neumann [6] and AAV [3] both used as a meter space the infinite dimensional space $L^2(\mathbf{R})$ of all complex-valued, square-integrable functions on the real line, known to physicists as the Hilbert space of a single spinless particle in one dimension. They both used a preparation Hamiltonian of the form $H := A \otimes P$, where P is the usual momentum operator defined by $(Pf)(q) := -i\, df/dq$, and their meter operator (which we called B) is the usual position operator Q defined by $Qf(q) := qf(q)$. AAV used an ϵ-dependent Gaussian as the square m^2 of their real meter function m (so that initially the meter position had a Gaussian distribution with a variance depending on a parameter ϵ). Thus their setup is algebraically quite similar to ours (and of course ours was inspired by theirs). There are some apparent differences which are discussed in detail in [1], but these turn out to be inessential.[8]

The calculations done above are rigorous for finite-dimensional S and M, and still algebraically correct in infinite dimensions. In the physics literature, such algebraic calculations are typically accepted as "proofs". If we relax mathematical rigor to this extent, we obtain from the above a very simple "proof" in the AAV context of the "usual" weak value $\Re(\langle f, As\rangle/\langle f, s\rangle)$ by taking

$$M := L^2(\mathbf{R}), \quad G := P, \quad B := Q, \quad m(q) := \left[\frac{1}{\sqrt{2\pi}} e^{-q^2/2}\right]^{1/2}. \tag{23}$$

This "proof" is deeply flawed because the starting equation (14),

$$e^{-i\epsilon(A\otimes G)}(s\otimes m) = s\otimes m - i\epsilon As\otimes Gm + O(\epsilon^2),$$

would probably be difficult to justify for our unbounded $G := P$. All of the "proofs" of the "usual" weak value formula which we have seen in the literature rely on uncontrolled approximations like this. In honesty, they should be called something like "algebraic motivations" instead of proofs.

If we *are* willing to accept uncritically such uncontrolled approximations, we can obtain arbitrary weak values in an AAV-type framework by taking B and m as in equation (23) and

$$G : P + \rho Q \quad \text{ with } \rho \neq 0 \text{ real}. \tag{24}$$

This results in

$$\langle m, BGm\rangle = \rho + \frac{i}{2} \quad , \tag{25}$$

so that when $\Im(\langle f, As\rangle/\langle f, s\rangle) \neq 0$, any "weak value" whatever can be obtained using the preparation Hamiltonian $H := A\otimes(P + \rho Q)$.

It is almost immediate that

$$e^{-iQ^2\rho/2} P e^{iQ^2\rho/2} = P + \rho Q \tag{26}$$

[8]We mention two here to forestall confusion for readers who may be consulting AAV. They use a preparation Hamiltonian $-A\otimes Q$ instead of our $A\otimes P$, and their meter observable is $I\otimes P$ instead of our $I\otimes Q$. Since the Fourier transform isomorphism takes Q to $-P$ and P to Q, this difference is immaterial.

Also, they use an ϵ-dependent meter state $m[\epsilon)](\cdot)$ which in our notation would be $m[\epsilon](q) := m(q\epsilon)\sqrt{\epsilon}$ in conjunction with an ϵ-independent preparation operator which in our notation would be $e^{iA\otimes P}$. This makes $\langle e^{iA\otimes P}(s\otimes m[\epsilon]), (P_f\otimes Q)e^{iA\otimes P}(s\otimes m[\epsilon])\rangle$ of constant order (i.e., $O(1)$ instead of $O(\epsilon)$ as in our formulation), so that our normalization by dividing by ϵ is unnecessary. The details are worked out in [1], resulting in the conclusion that their setup is algebraically equivalent to ours.

because for any $g \in L^2(\mathbf{R})$, $(e^{iQ^2\rho/2}g)(q) = e^{iq^2\rho/2}g(q)$ and $P := -i\,d/dq$.[9] In other words, our preparation Hamiltonian $P + \rho Q$ is formally (i.e., algebraically, ignoring analytical subtleties) unitarily equivalent to AAV's. This unitary equivalence carries the Gaussian meter state $m(q)$ (which is effectively that used by AAV) into

$$q \mapsto e^{-iq^2\rho/2}m(q) \quad , \tag{27}$$

which still defines a Gaussian probability distribution on position space.

A rigorous derivation of the "usual" weak value in the AAV framework (i.e. meter space $L^2(\mathbf{R})$ and using the AAV Hamiltonian and meter state was given in [1]), along with an extension of that argument which yields a rigorous proof that weak values are not unique even using the AAV Hamiltonian $A \otimes P$ (but with the slightly different meter state (27)).

6 Remarks

1. Definition 1 of "weak measurement protocol" assumed a "reproducible" state s of S, i.e., a state for which an arbitrary number of copies are available.[10] All weak measurement schemes known to us require this assumption. This is because for very weak coupling between the system S of interest and the meter system, a large number of meter measurements may be needed to obtain a reliable average.

   In finite dimensions, a reproducible state may be considered "known" in the sense that its components with respect to a given basis can be estimated to arbitrary accuracy by quantum tomography ([2], pp. 389ff). But if s is known, then so is $\langle s, As\rangle$, and one wonders what is the point of finding a weak measurement protocol to measure $\langle s, As\rangle$. It is true that such a protocol can perform the measurement with negligible effect on the copies of s used in the measurement, but it's hard to see how this could be useful when an arbitrary number of copies of s were available from the start.

2. Once one realizes that it is either tautological or incorrect to identify the normalized conditional expectation $NE(B|f)$ of $I \otimes B$ with the "expectation of A in state s conditional on postselection to f", the concept of "weak measurement" seems to collapse. What is left?

   When the denominator $\langle f, s\rangle$ of the traditional weak value $\Re(\langle f, As\rangle/\langle f, s\rangle)$ is small, the weak value can be very large relative to the norm of A. This is sometimes considered as an "amplification" effect. Assuming that the experimental procedure faithfully implements the mathematics leading to

[9] We say "almost" because a rigorous verification would require careful specification of the domain of P, which we have not discussed.

[10] This does not contradict the no-cloning theorem because is is not required that an arbitrary state can be copied, only that we have a device which can make any number of copies of some particular state in which we are interested.

that weak value, the "amplification" is real in the sense that the normalized conditional (i.e., on successful postselection) expectation of the meter can greatly exceed all eigenvalues of A. Reference [5] reports such amplification. (For an expository account, see [4].) However, in a general context, it seems unclear precisely what is being "amplified", if "amplification" is considered a linear process as usual. It cannot be $\langle s, As\rangle$ because the traditional weak value $\Re(\langle f, As\rangle/\langle f, s\rangle)$ is not linear in $\langle s, As\rangle$. Indeed, one could have $\langle s, As\rangle = 0$ with the weak value $\Re(\langle f, As\rangle/\langle f, s\rangle)$ nonzero.

3. Nearly all of the "weak value" literature (with the notable exception of a recent paper [7] of Jozsa) presents the traditional "weak value" $\Re(\langle f, As\rangle/\langle f, s\rangle)$ as if it were the only theoretical possibility, and as if it were experimentally inevitable. Even the Jozsa paper does not comment on the implications of the nonuniqueness of weak values.

   In view of the nonuniqueness of weak values, claims of the experimental inevitability of the traditional weak value should be carefully scrutinized. It seems strange that papers describing complicated experiments to measure weak values generally ignore this crucial point. It should not be taken for granted that the AAV Hamiltonian and meter state can be assumed without detailed justification.

7 Summary

We have defined "weak measurement" and "weak value" of a quantum observable, and have given a rigorous proof in a finite dimensional context that weak values of a quantum observable A in a state s of a quantum system S are not unique. This implies that weak values need not be intrinsic to the system S being "weakly" measured: in general they may depend also on details of the meter system (such as the meter state).

Most of the "weak value" literature presents the traditional weak value $\Re(\langle f, As\rangle/\langle f, s\rangle)$ as if it were experimentally inevitable, without mentioning the possibility of other "weak values". We suspect that this may arise from the logical fallacy of identifying the (normalized) expectation of a "meter observable" conditional on postselection to $f \in S$ with the expectation of A conditional on postselection to f. The (normalized) expectation of the meter observable does equal the expectation of A, but there is no reason that the conditional expectations should be equal.

Traditional "weak values" are usually associated with a particular infinite dimensional context introduced in AAV [3]. Though the analysis given in this paper is rigorous only for finite dimensional systems, it can be rigorously extended to the AAV context, yielding multiple weak values in that context; details are given in [1].

The nonuniqueness of "weak values" suggests that any claims of the inevitability of the "traditional" weak value $\Re(\langle f, As\rangle/\langle f, s\rangle)$ in experimental weak measurements will probably have to be based on arguments for the universality of something close to the precise AAV setup (e.g., Hamiltonian, meter observable, and meter state). Since there seems to exist no argument in the literature that this precise setup can be realized in *any* experimental situation, much less be inevitable in all, such an argument would probably have to break new ground.

8 Afterword

I thank an anonymous referee for drawing my attention to an interesting paper of R. Jozsa [7] which motivates (formally, using uncontrolled approximations) a "Theorem" relating weak values to the real and imaginary parts of the complex AAV weak value $\langle f, As\rangle/\langle f, s\rangle$, along with the mean and variance of the meter observable, assuming that the meter state satisfies a Schroedinger equation before the preparation Hamiltonian is applied.

In our notation (and under our assumption that $\langle m, Qm\rangle = 0$) this Theorem states that (recall that $NE(Q|f)$ is the normalized expectation of the meter observable conditional on postselection to $f \in S$)

$$NE(Q|f) = \Re\frac{\langle f, As\rangle}{\langle f, s\rangle} + \Im\frac{\langle f, As\rangle}{\langle f, s\rangle}k\left.\frac{d\langle m, Q^2m\rangle}{dt}\right|_{t=0^-} \quad , \tag{28}$$

where Q is the meter observable of our Section 5 (i.e., $Qf(q) := qf(q)$ for $f \in L^2(\mathbf{R})$) and k is the mass of the meter pointer.[11] (Because the meter state m is assumed to satisfy a Schroedinger equation, both it and $\langle m, Qm\rangle$ are time-dependent.)

If we assume that the coefficient of $\Im(\langle f, As\rangle/\langle f, s\rangle)$ can be arbitrary (as seems reasonable and probably provable under mild additional hypotheses), this is like our (20) for this particular setup (i.e., meter state assumed to satisfy a Schroedinger equation). In particular, this motivates the existence of arbitrary weak values for a given observable A. However, [7] does not comment on the implications of the fact that weak values are not unique.

I also thank that referee for pointing out a potential ambiguity in the original exposition, which has been corrected in this version.

Another referee objected to the paper's suggestion that "we think that typical characterizations of 'weak values' involve an implicit logical fallacy". Since this is a personal opinion which is clearly identified as such, I saw no reason to excise it from this later version.

Most presentations of weak values in the literature are so vaguely written that is is difficult to discern precisely what the authors might have meant. I think it would be very difficult to explicitly and convincingly motivate the concept of "weak value" without comitting some such logical fallacy. The exposition required some such motivation, so I made my best guess, and it still seems to me the best guess. If some reader thinks that guess obviously wrong, he or she is

[11] The reader should be warned that the proof of the stated "Theorem" giving this relation seems to require an unstated hypothesis that a boundary term in a partial integration (his equation (14)) can be dropped; otherwise the coefficient of $\Im(\langle f, As\rangle/\langle f, s\rangle)$ in (28) will not necessarily be as stated.

invited to provide an alternate motivation which is clear, correct, and consistent with the traditional literature.

More information about the submission history of this paper along with all referees' reports and comments on them can be found on my website, www.math.umb.edu/∼sp in the "papers" page.

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