Part A.  
Do all four of the problems.

1. Find the first derivative of each function.
   \[ y = \arcsin(x/3) \quad b) \quad y = \ln(\sin(2x)) \quad c) \quad y = x \tan(1 + bx) \]
   \[ \text{Ans. a) } (9 - x^2)^{-1/2} \quad b) \quad 2 \cot 2x \quad c) \quad \tan(1 + bx) + \frac{bx}{(1 + bx)^2 + 1} \]

2. Evaluate the limits. Explain your answers.
   \[ a) \lim_{x \to 0^+} \frac{\ln x}{x} \quad b) \lim_{x \to 0} \frac{\ln x}{x} \quad c) \lim_{x \to 0^+} x^x \]
   \[ \text{Ans. a) } -\infty \quad b) \quad 0 \quad c) \quad 1 \]

3. Integrate.
   \[ a) \int \frac{\sqrt{x^2 - 1}}{x} \, dx \quad b) \int e^x \sin 2x \, dx \quad c) \int \frac{1}{\sqrt{6x - x^2}} \, dx \]
   \[ \text{Ans. a) } 1 - \frac{x}{4} \quad b) \frac{1}{2} e^x (\sin 2x - 2 \cos 2x) + C \quad c) \arcsin\left(\frac{x - 3}{3}\right) + C \]

4. Evaluate the improper integral or show it diverges.
   \[ a) \int_0^\infty te^{-t} \, dt \quad b) \int_0^\infty \frac{1}{(x + 2)(x + 3)} \, dt \quad c) \int_{-\infty}^\infty u^2 e^{-u^2} \, du \]
   \[ \text{Ans. a) } \frac{1}{1} \quad b) \quad -\frac{1}{1} \quad c) \quad \text{div} \]

Part B.  
Do five of the six problems.

5. A curve is shaped so that its slope at each point is three times the y-coordinate of the point. The curve passes through the point (1, 3). Find the equation of the curve. \[ \text{Ans. } y = Ce^{3x}, \text{ and } 3 = Ce^3. \text{ So } y = 3e^{3x-3}. \]

6. According to Coulomb's Law, a proton attracts an electron with a force \( F = kr^{-2} \), where \( k \) is a positive constant (depending on the units of measurement) and \( r \) is the distance between the particles. Suppose the electron is at distance \( r = 1/2 \) from the proton and we move it to a distance \( r = 3 \). What work must we do to accomplish this?
   Show how to approximate the work by a Riemann sum, explaining your choice of coordinate and your expression for the "element" \( \Delta W \) of work being summed. Then express the work as an integral and evaluate it (the answer will involve \( k \)).
   \[ \text{Ans. } \Delta W = F \Delta r = kr^{-2} \Delta r, \text{ so } W = \frac{3}{14} \Delta W = 5k/3 \]

7. The probability density for decay of Strontium 90 is \( f(t) = \begin{cases} (1/40)e^{-t/40}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0, \end{cases} \) with \( t \) in years.
   What is the probability that a given atom of Strontium 90
   \[ a) \text{ will decay in five years?} \quad b) \text{ will survive for twenty-five years?} \]
   \[ \text{Ans. a) } 1 - e^{-1/8} \quad e^{-5/8} \]

8. The region bounded by \( x = 0, \) \( x = 1, \) \( y = x \) and \( y = e^x \) is revolved around the \( y \)-axis. Compute the volume of the resulting solid, using shells:
   Sketch the solid, showing clearly your choice of coordinates, and indicate the "element" \( \Delta V \) of volume being summed. Then evaluate the integral to find the volume.
   \[ \text{Ans. Shell has radius } x, \text{ height } e^x - x \text{ and wall thickness } \Delta x, \text{ so the element of volume } V \text{ is } \Delta V = (e^x - x)(2\pi x) \Delta x. \text{ Then } V = \int_0^1 dV = 4\pi/3. \]

The exam continues on the next side.
9. a) Find polar coordinates \( r \) and \( \theta \) for the point \((2, 2)\), first with \( r > 0 \), and then with \( r < 0 \).

   \( r = \frac{27}{6 - 3 \cos \theta} \)  

   \( \theta = \frac{\pi}{4} \)  

   Ans. \(( \pm 2\sqrt{2}, \pi/4)\)

b) Sketch the graph of the polar equation \( r = \frac{27}{6 - 3 \cos \theta} \). Identify it as ellipse, parabola or hyperbola. Give Cartesian coordinates of the center, vertices and foci.

   Ans. ellipse, \( e = 1/2 \); vertices \((9, 0), (-3, 0)\)

10. a) Let \( w = \frac{1}{-1 + 2i} \). Express \( w \) in each of the given forms:

   i) \( w = x + iy \), \( x = \frac{1}{5}, y = \frac{2}{5} \)

   ii) \( w = re^{i\theta} \) (with \( r > 0 \)).

   \( r = \sqrt{5}, \theta = \arctan(2) \)

   Ans. i) \( w = (-1 - 2i)/5 \); ii) \( w = \frac{\sqrt{5}}{5} e^{i \arctan(2)} \)

b) Find all solutions to \( z^4 + 4 = 0 \), and plot the solutions in a complex plane.

   \( z = \sqrt{2} \exp(i \pi k/4), k = 1, 3, 5, 7 \)

Part C. Do three of the four problems.

11. a) For each sequence, evaluate its limit or show it diverges.

   i) \( \lim_{n \to \infty} \left( \frac{n - 1}{n} \right)^{1001} \)

   ii) \( \lim_{n \to \infty} \left( \frac{n - 1}{n} \right)^n \)

   Ans. i) 1; ii) 1/e

b) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Identify the convergence tests you use.

   i) \( \sum_{n=1}^{\infty} (-1)^n \sin \left( \frac{\pi}{n} \right) \)

   ii) \( \sum_{n=1}^{\infty} (-1)^n ne^{-n} \)

   Ans. i) absil series div by lim comp with \( \Sigma \frac{1}{n^2} \), series conv by alt, \( \cdot \) cond conv; ii) absil conv by integral or ratio test

12. Test the series for convergence. Tell the convergence tests you use, and how you use them.

   a) \( \sum_{n=1}^{\infty} \frac{n^3}{5^n} \)

   b) \( \sum_{n=1}^{\infty} \left( \frac{n}{3n + 1} \right)^n \)

   c) \( \sum_{n=1}^{\infty} \ln \left( \frac{n}{3n + 1} \right) \)

   d) \( \sum_{n=1}^{\infty} \frac{1}{1 + n^2} \)

   e) \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \)

   Ans. a) conv, Ratio test; b) conv, Root test; c) term\( \to 0 \), div; d) absil conv, compare \( \sum 1/n^2 \); e) conv, Integral test

13. a) Use series methods to compute the numbers.

   i) \( \lim_{x \to 0} \frac{\sin x}{x} \)

   ii) \( \int_0^1 \cos(x^2) \, dx \) with |error| < 10^{-3}

   Ans. i) 1; ii) \( 1 - \frac{1}{5 \cdot 2!} + \frac{1}{9 \cdot 4!} \)

b) Let \( F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} (x - 1)^n \).

   i) Represent \( F'(x) \) as an infinite series.

   \( F'(x) = \sum_{n=1}^{\infty} \frac{1}{n} (x - 1)^{n-1} \)

   ii) Find the intervals of convergence of \( F \) and \( F' \). Are they the same?

   Ans. \([0, 2]\) for \( F \), \([0, 2) \) for \( F' \); not the same

14. Let \( h(x) = \ln \left( \frac{1 + x}{1 - x} \right) \), for \(-1 < x < 1\). We’ll use \( h \) to compute \( \ln 2 \). Show first that \( \ln 2 = h(1/3) \).

   a) Show next by using partial fractions that \( h(x) = 2 \int_0^x \frac{dt}{1 - t^2} \). Ans. use that \( \frac{2}{1 - t^2} = \frac{1}{1 - t} + \frac{1}{1 + t} \)

   b) Find the Maclaurin series for \( h \) and its radius of convergence, by using a geometric series.

   \( h(x) = 2 \sum_{n=0}^{\infty} \frac{1}{1 + t^2} = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right) \); \( R = 1 \)

   c) Finally, obtain the expression

   \( \ln 2 = 2 \left( \frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \cdots \right) \)

End of Sample Final Exam. ! The Exam may include problems which are not exactly like any of the problems here.