The Difference Quotient, and the slope of the tangent line to a curve. Name ___________ Section ____

Objectives: (1) to understand and calculate the difference quotient.
(2) to use the difference quotient to find the slope of the tangent line to a function curve.

Let $f(x)$ be a function.

A secant line to $f(x)$ is a line passing through two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ of $f(x)$.

The slope of the secant line is $\Delta y/\Delta x$, or:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$  

We want to keep $x_1$ fixed, and squeeze $x_2$ closer and closer to $x_1$.

To emphasize this idea, we rewrite the slope of the secant line substituting $x$ for $x_1$, and $(x+h)$ for $x_2$. See the picture at the right.

Since $(x+h) - x = h$, the denominator in the slope expression is $h$. The expression is called the **difference quotient**, and is written:

$$\frac{f(x+h) - f(x)}{h}.$$  

**Examples.**

**Example 1.** $f(x) = 3x$. This is a linear equation. Given any two points on this line, we know the slope is 3.

Calculating using the difference quotient: slope = $\frac{f(x+h) - f(x)}{h} = \frac{3(x+h) - 3x}{h} = 3$.

**Example 2.** $f(x) = 4x^2$. This is a quadratic equation. The slope of the secant line depends on the point $x$, and also on the distance $h$ between $x$ and $x_2$. (see the picture above: the red line is the secant line).

Calculating using the difference quotient, slope = $\frac{f(x+h) - f(x)}{h} = \frac{4(x^2+2xh+h^2) - 4x^2}{h} = \frac{8xh+4h^2}{h} = 8x+4h$.

**What we are really interested in is the slope of the secant line when the two points are very, very close.** We can’t set the value of $h$ to zero, because then the difference quotient becomes $(0/0)$ which is undefined. However, we do a limiting process. We let $h$ get closer and closer to zero, and try to find out how the slope of the secant line changes. **If the slope of the secant line has a limiting value**, then that value will approach the slope of the **tangent line** (the line that passes through $(x,f(x))$ and which has the same direction as the function curve at that point).

Back to example 2. As the value of $h$ approaches zero, then the value of the slope of the secant line approaches $8x$, since $4h$ gets very small. So the tangent line to the parabola $f(x)=4x^2$ at the point $(x,x^2)$ has slope $m=8x$. For example, at $(x,y)=(-2,16)$, then the slope of our parabola is $8(-2)=-16$ at $(-2,16)$.

This value, the slope of the tangent line to a function $y = f(x)$, at the point $(x, f(x))$, is called **$Df(x)$**, or the **derivative of $f(x)$ at $x$**.

The derivative is calculated in two steps:

**Step 1.** Find the difference quotient.

**Step 2.** Find the limit, if it exists, of the difference quotient, as $h$ approaches zero. (symbol: $h \to 0$).

This limit process is a little bit tricky, and requires some mathematical machinery. Right now, we want to calculate difference quotients, and use some common sense to evaluate the limit value.

**Problems.** Find the difference quotient, and, if possible, the derivative, of each function:

1. $f(x) = 3x^2$
2. $f(x) = 2x-5$
3. $g(x) = 2x^2 - 1$
4. $k(x) = x^2 - 5x$
5. $f(x) = e^x$
6. $f(x) = x^2 - 2x + 1$
7. $f(x) = x^3$
8. $f(x) = (1/2) x^2 - 3$
9. $p(x) = 1/x$
10. $f(x) = 1/(x-2)$
11. $f(x) = (x+3)/(x-1)$
12. $f(x) = \cos(x)$ [this one is a bit tricky]