1.3 On the Solutions of Linear Systems; Matrix Algebra

7. Consider the vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) in \( \mathbb{R}^2 \) shown in the accompanying sketch. How many solutions \( x, y \) does the system

\[
x \vec{v}_1 + y \vec{v}_2 = \vec{v}_3
\]

have? Argue geometrically.

8. Consider the vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \) in \( \mathbb{R}^2 \) shown in the accompanying sketch. Arguing geometrically, find two solutions \( x, y, z \) of the linear system

\[
x \vec{v}_1 + y \vec{v}_2 + z \vec{v}_3 = \vec{v}_4.
\]

How do you know that this system has, in fact, infinitely many solutions?

9. Write the system

\[
\begin{align*}
x + 2y + 3z &= 1 \\
4x + 5y + 6z &= 4 \\
7x + 8y + 9z &= 9
\end{align*}
\]

in matrix form.

**Compute the dot products in Exercises 10 through 12 (if the products are defined).**

10. \[
\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\]

11. \[
\begin{bmatrix} 1 & 9 & 9 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \end{bmatrix}
\]

12. \[
\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}
\]

**Compute the products \( A\vec{x} \) in Exercises 13 through 15 using paper and pencil. In each case, compute the product two ways: in terms of the columns of \( A \) (Theorem 1.3.8) and in terms of the rows of \( A \) (Definition 1.3.7).**

13. \[
\begin{bmatrix} 1 & 2 & 7 \\ 3 & 4 & 11 \end{bmatrix}
\]

14. \[
\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}
\]

15. \[
\begin{bmatrix} 2 & 3 & 4 \\ 5 \\ 6 & 7 & 8 \end{bmatrix}
\]

**Compute the products \( A\vec{x} \) in Exercises 16 through 19 using paper and pencil (if the products are defined).**

16. \[
\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}
\]

17. \[
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}
\]

18. \[
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}
\]

19. \[
\begin{bmatrix} -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\]

20. a. Find \[
\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ 3 & 1 \\ 0 & -1 \end{bmatrix}
\]

b. Find \[
\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix}
\]

21. Use technology to compute the product

\[
\begin{bmatrix} 1 & 7 & 8 \\ 1 & 2 & 9 \\ 1 & 5 & 1 & 5 \\ 1 & 6 & 4 & 8 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \\ 5 \\ 6 \end{bmatrix}
\]

22. Consider a linear system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

23. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

24. Let \( A \) be a \( 4 \times 4 \) matrix, and let \( \vec{b} \) and \( \vec{c} \) be two vectors in \( \mathbb{R}^4 \). We are told that the system \( A\vec{x} = \vec{b} \) has a unique solution. What can you say about the number of solutions of the system \( A\vec{x} = \vec{c} \)?

25. Let \( A \) be a \( 4 \times 4 \) matrix, and let \( \vec{b} \) and \( \vec{c} \) be two vectors in \( \mathbb{R}^4 \). We are told that the system \( A\vec{x} = \vec{b} \) is inconsistent. What can you say about the number of solutions of the system \( A\vec{x} = \vec{c} \)?

26. Let \( A \) be a \( 4 \times 3 \) matrix, and let \( \vec{b} \) and \( \vec{c} \) be two vectors in \( \mathbb{R}^4 \). We are told that the system \( A\vec{x} = \vec{b} \) has a unique solution. What can you say about the number of solutions of the system \( A\vec{x} = \vec{c} \)?

27. If the rank of a \( 4 \times 4 \) matrix \( A \) is 4, what is \( \text{rref}(A) \)?

28. If the rank of a \( 5 \times 3 \) matrix \( A \) is 3, what is \( \text{rref}(A) \)?

29. Find a diagonal matrix \( A \) such that \( A\vec{x} = \vec{y} \).

30. Find a matrix \( A \) of rank 1 such that \( A\vec{x} = \vec{y} \).

31. Find an upper triangular matrix \( A \) such that \( A\vec{x} = \vec{y} \).
where all the entries of \( A \) on and above the diagonal are nonzero.

32. Find a matrix \( A \) with all nonzero entries such that \( A \tilde{x} = \tilde{y} \).

33. Let \( A \) be the \( n \times n \) matrix with all 1’s on the diagonal and all 0’s above and below the diagonal. What is \( A \tilde{x} \), where \( \tilde{x} \) is a vector in \( \mathbb{R}^n \)?

34. We define the vectors
\[
\tilde{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
in \( \mathbb{R}^3 \).

a. For
\[
A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix},
\]
compute \( A \tilde{e}_1, A \tilde{e}_2, \) and \( A \tilde{e}_3 \).

b. If \( B \) is an \( n \times 3 \) matrix with columns \( \tilde{v}_1, \tilde{v}_2, \) and \( \tilde{v}_3 \), what are \( B \tilde{e}_1, B \tilde{e}_2, \) and \( B \tilde{e}_3 \)?

35. In \( \mathbb{R}^m \), we define
\[
\tilde{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \quad \text{← \( i \)th component.}
\]

If \( A \) is an \( n \times m \) matrix, what is \( A \tilde{e}_i \)?

36. Find a \( 3 \times 3 \) matrix \( A \) such that
\[
A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},
\]
and
\[
A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.
\]

37. Find all vectors \( \tilde{x} \) such that \( A \tilde{x} = \tilde{b} \), where
\[
A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.
\]

a. Using technology, generate a random \( 3 \times 3 \) matrix \( A \). (The entries may be either single-digit integers or numbers between 0 and 1, depending on the technology you are using.) Find \( \text{rref}(A) \). Repeat this experiment a few times.

b. What does the reduced row-echelon form of most \( 3 \times 3 \) matrices look like? Explain.

38. a. Find \( \text{rref}(A) \) of the system \( A \tilde{x} = \tilde{b} \).

b. What does the reduced row-echelon form of most \( 3 \times 3 \) matrices look like? Explain.

39. Repeat Exercise 38 for \( 3 \times 4 \) matrices.

40. Repeat Exercise 38 for \( 4 \times 3 \) matrices.

41. How many solutions do most systems of three linear equations with three unknowns have? Explain in terms of your work in Exercise 38.

42. How many solutions do most systems of three linear equations with four unknowns have? Explain in terms of your work in Exercise 39.

43. How many solutions do most systems of four linear equations with three unknowns have? Explain in terms of your work in Exercise 40.

44. Consider an \( n \times m \) matrix \( A \) with more rows than columns \( (n > m) \). Show that there is a vector \( \tilde{b} \) in \( \mathbb{R}^n \) such that the system \( A \tilde{x} = \tilde{b} \) is inconsistent.

45. Consider an \( n \times m \) matrix \( A \), a vector \( \tilde{x} \) in \( \mathbb{R}^m \), and a scalar \( k \). Show that
\[
A(k \tilde{x}) = k (A \tilde{x}).
\]

46. Find the rank of the matrix
\[
\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix},
\]
where \( a, d, \) and \( f \) are nonzero, and \( b, c, \) and \( e \) are arbitrary numbers.

47. A linear system of the form
\[
A \tilde{x} = \tilde{0}
\]
is called homogeneous. Justify the following facts:

a. All homogeneous systems are consistent.

b. A homogeneous system with fewer equations than unknowns has infinitely many solutions.

c. If \( \tilde{x}_1 \) and \( \tilde{x}_2 \) are solutions of the homogeneous system \( A \tilde{x} = \tilde{0} \), then \( \tilde{x}_1 + \tilde{x}_2 \) is a solution as well.

d. If \( \tilde{x} \) is a solution of the homogeneous system \( A \tilde{x} = \tilde{0} \) and \( k \) is an arbitrary constant, then \( k \tilde{x} \) is a solution as well.

48. Consider a solution \( \tilde{x}_1 \) of the linear system \( A \tilde{x} = \tilde{b} \). Justify the facts stated in parts (a) and (b):

a. If \( \tilde{x}_h \) is a solution of the system \( A \tilde{x} = \tilde{0} \), then \( \tilde{x}_1 + \tilde{x}_h \) is a solution of the system \( A \tilde{x} = \tilde{b} \).

b. If \( \tilde{x}_2 \) is another solution of the system \( A \tilde{x} = \tilde{b} \), then \( \tilde{x}_2 - \tilde{x}_1 \) is a solution of the system \( A \tilde{x} = \tilde{0} \).

c. Now suppose \( A \) is a \( 2 \times 2 \) matrix. A solution vector \( \tilde{x}_1 \) of the system \( A \tilde{x} = \tilde{b} \) is shown in the accompanying figure. We are told that the solutions of the system \( A \tilde{x} = \tilde{0} \) form the line shown in the sketch. Draw the line consisting of all solutions of the system \( A \tilde{x} = \tilde{b} \).
54. Consider two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^3$ that are not parallel. Which vectors in $\mathbb{R}^3$ are linear combinations of $\vec{v}_1$ and $\vec{v}_2$? Describe the set of these vectors geometrically. Include a sketch in your answer.

55. Is the vector
\[
\begin{bmatrix}
7 \\
8 \\
9
\end{bmatrix}
\]
a linear combination of
\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\text{ and }
\begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix}?
\]

56. Is the vector
\[
\begin{bmatrix}
30 \\
-1 \\
38 \\
56 \\
62
\end{bmatrix}
\]
a linear combination of
\[
\begin{bmatrix}
1 \\
7 \\
1 \\
9 \\
4
\end{bmatrix}
\text{ and }
\begin{bmatrix}
5 \\
6 \\
3 \\
2 \\
8
\end{bmatrix}?
\]

57. Express the vector
\[
\begin{bmatrix}
7 \\
11
\end{bmatrix}
\]
as the sum of a vector on the line $y = 3x$ and a vector on the line $y = x/2$.

58. For which values of the constants $b$ and $c$ is the vector
\[
\begin{bmatrix}
b \\
c
\end{bmatrix}
\]
a linear combination of
\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\text{ and }
\begin{bmatrix}
-1 \\
2 \\
-3
\end{bmatrix}?
59. For which values of the constants \( c \) and \( d \) is \[
\begin{bmatrix}
1 & 2 \\
1 & 3 \\
1 & 4 \\
\end{bmatrix}
\]
a linear combination of \[
\begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}
\text{ and } \begin{bmatrix}
2 \\
3 \\
4 \\
\end{bmatrix}?
\]

60. For which values of the constants \( a, b, c, \) and \( d \) is \[
\begin{bmatrix}
a \\
b \\
c \\
d \\
\end{bmatrix}
a linear combination of \[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
3 & 4 & 0 & 0 \\
0 & 5 & 0 & 0 \\
\end{bmatrix}?
\]

61. For which values of the constant \( c \) is \[
\begin{bmatrix}
1 & c \\
2 & c^2 \\
4 & 9 \\
\end{bmatrix}
a linear combination of \[
\begin{bmatrix}
1 \\
1 \\
2 \\
4 \\
\end{bmatrix}\text{ and } \begin{bmatrix}
1 \\
1 \\
3 \\
9 \\
\end{bmatrix}?
\]

62. For which values of the constant \( c \) is \[
\begin{bmatrix}
1 & c \\
1 & c^2 \\
\end{bmatrix}
a linear combination of \[
\begin{bmatrix}
a \\
1 \\
\end{bmatrix}\text{ and } \begin{bmatrix}
b \\
1 \\
\end{bmatrix},\text{ where } a \text{ and } b \text{ are arbitrary constants?}
\]

In Exercises 63 through 68, consider the vectors \( \vec{v} \) and \( \vec{w} \) in the accompanying figure.

63. Give a geometrical description of the set of all vectors of the form \( \vec{v} + c \vec{w} \), where \( c \) is an arbitrary real number.

64. Give a geometrical description of the set of all vectors of the form \( \vec{v} + c \vec{w} \), where \( 0 \leq c \leq 1 \).

65. Give a geometrical description of the set of all vectors of the form \( a \vec{v} + b \vec{w} \), where \( 0 \leq a \leq 1 \) and \( 0 \leq b \leq 1 \).

66. Give a geometrical description of the set of all vectors of the form \( a \vec{v} + b \vec{w} \), where \( a + b = 1 \).

67. Give a geometrical description of the set of all vectors of the form \( a \vec{v} + b \vec{w} \), where \( 0 \leq a, 0 \leq b, \) and \( a + b \leq 1 \).

68. Give a geometrical description of the set of all vectors \( \vec{w} \) in \( \mathbb{R}^2 \) such that \( \vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} \).

69. Solve the linear system
\[
\begin{align*}
y + z &= a \\
x + z &= b \\
x + y &= c
\end{align*}
\]
where \( a, b, \) and \( c \) are arbitrary constants.

70. Let \( A \) be the \( n \times n \) matrix with 0’s on the main diagonal, and 1’s everywhere else. For an arbitrary vector \( \vec{b} \) in \( \mathbb{R}^n \), solve the linear system \( A\vec{x} = \vec{b} \), expressing the components \( x_1, \ldots, x_n \) of \( \vec{x} \) in terms of the components of \( \vec{b} \). See Exercise 69 for the case \( n = 3 \).

Chapter One Exercises

TRUE OR FALSE?\(^1\)

Determine whether the statements that follow are true or false, and justify your answer.

1. If \( A \) is an \( n \times n \) matrix and \( \vec{x} \) is a vector in \( \mathbb{R}^n \), then the product \( A\vec{x} \) is a linear combination of the columns of matrix \( A \).

\(^1\)We will conclude each chapter (except for Chapter 9) with some true–false questions, over 400 in all. We will start with a group of about 10 straightforward statements that refer directly to definitions and theorems given in the chapter. Then there may be some computational exercises, and the remaining ones are more conceptual, calling for independent reasoning. In some chapters, a few of the problems toward the end can be quite challenging. Don’t expect a balanced coverage of all the topics; some concepts are better suited for this kind of questioning than others.

2. If vector \( \vec{u} \) is a linear combination of vectors \( \vec{v} \) and \( \vec{w} \), then we can write \( \vec{u} = a\vec{v} + b\vec{w} \) for some scalars \( a \) and \( b \).

3. Matrix \[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
is in reduced row-echelon form.

4. A system of four linear equations in three unknowns is always inconsistent.

5. There exists a \( 3 \times 4 \) matrix with rank 4.

6. If \( A \) is a \( 3 \times 4 \) matrix and vector \( \vec{v} \) is in \( \mathbb{R}^4 \), then vector \( A\vec{v} \) is in \( \mathbb{R}^3 \).

7. If the \( 4 \times 4 \) matrix \( A \) has rank 4, then any linear system with coefficient matrix \( A \) will have a unique solution.
8. Find the inverse of the linear transformation 
   \[ y_1 = x_1 + 7x_2 \]
   \[ y_2 = 3x_1 + 20x_2. \]

In Exercises 9 through 12, decide whether the given matrix is invertible. Find the inverse if it exists. In Exercise 12, the constant \( k \) is arbitrary.

9. \[
   \begin{bmatrix}
   2 & 3 \\
   6 & 9
   \end{bmatrix}
   \]

10. \[
   \begin{bmatrix}
   1 & 2 \\
   4 & 9
   \end{bmatrix}
   \]

11. \[
   \begin{bmatrix}
   1 & 2 \\
   3 & 9
   \end{bmatrix}
   \]

12. \[
   \begin{bmatrix}
   1 & k \\
   0 & 1
   \end{bmatrix}
   \]

13. Prove the following facts:
   a. The \( 2 \times 2 \) matrix
      \[
      A = \begin{bmatrix}
      a & b \\
      c & d
      \end{bmatrix}
      \]
      is invertible if and only if \( ad - bc \neq 0 \). \( \text{Hint: Consider the cases } a \neq 0 \text{ and } a = 0 \text{ separately.} \)
   b. If
      \[
      \begin{bmatrix}
      a & b \\
      c & d
      \end{bmatrix}
      \]
      is invertible, then
      \[
      \begin{bmatrix}
      a & b \\
      c & d
      \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix}
      d & -b \\
      -c & a
      \end{bmatrix}.
      \]
      [The formula in part (b) is worth memorizing.]

14. a. For which values of the constant \( k \) is the matrix
       \[
       \begin{bmatrix}
       2 & 3 \\
       5 & k
       \end{bmatrix}
       \]
       invertible?
   b. For which values of the constant \( k \) are all entries of
       \[
       \begin{bmatrix}
       2 & 3 \\
       5 & k
       \end{bmatrix}^{-1}
       \]
       integers?
       See Exercise 13.

15. For which values of the constants \( a \) and \( b \) is the matrix
       \[
       A = \begin{bmatrix}
       a & -b \\
       b & a
       \end{bmatrix}
       \]
       invertible? What is the inverse in this case? See Exercise 13.

Give a geometric interpretation of the linear transformations defined by the matrices in Exercises 16 through 23. Show the effect of these transformations on the letter \( L \) considered in Example 5. In each case, decide whether the transformation is invertible. Find the inverse if it exists, and interpret it geometrically. See Exercise 13.

16. \[
   \begin{bmatrix}
   3 & 0 \\
   0 & 3
   \end{bmatrix}
   \]

17. \[
   \begin{bmatrix}
   -1 & 0 \\
   0 & -1
   \end{bmatrix}
   \]

18. \[
   \begin{bmatrix}
   0.5 & 0 \\
   0 & 0.5
   \end{bmatrix}
   \]

19. \[
   \begin{bmatrix}
   1 & 0 \\
   0 & 0
   \end{bmatrix}
   \]

20. \[
   \begin{bmatrix}
   0 & 1 \\
   1 & 0
   \end{bmatrix}
   \]

21. \[
   \begin{bmatrix}
   0 & 1 \\
   -1 & 0
   \end{bmatrix}
   \]

22. \[
   \begin{bmatrix}
   1 & 0 \\
   0 & -1
   \end{bmatrix}
   \]

23. \[
   \begin{bmatrix}
   0 & 2 \\
   -2 & 0
   \end{bmatrix}
   \]

Consider the circular face in the accompanying figure. For each of the matrices \( A \) in Exercises 24 through 30, draw a sketch showing the effect of the linear transformation \( T(\vec{x}) = A\vec{x} \) on this face.

24. \[
   \begin{bmatrix}
   0 & -1 \\
   1 & 0
   \end{bmatrix}
   \]

25. \[
   \begin{bmatrix}
   2 & 0 \\
   0 & 2
   \end{bmatrix}
   \]

26. \[
   \begin{bmatrix}
   0 & 1 \\
   1 & 0
   \end{bmatrix}
   \]

27. \[
   \begin{bmatrix}
   1 & 0 \\
   0 & -1
   \end{bmatrix}
   \]

28. \[
   \begin{bmatrix}
   0 & 0 \\
   0 & 1
   \end{bmatrix}
   \]

29. \[
   \begin{bmatrix}
   -1 & 0 \\
   0 & -1
   \end{bmatrix}
   \]

30. \[
   \begin{bmatrix}
   0 & 0 \\
   0 & 1
   \end{bmatrix}
   \]

31. In Chapter 1, we mentioned that an old German bill shows the mirror image of Gauss's likeness. What linear transformation \( T \) can you apply to get the actual picture back?

32. Find an \( n \times n \) matrix \( A \) such that \( A\vec{x} = 3\vec{x} \), for all \( \vec{x} \) in \( \mathbb{R}^n \).

33. Consider the transformation \( T \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) that rotates any vector \( \vec{x} \) through an angle \( 45^\circ \) in the counterclockwise direction, as shown in the following figure:

![Diagram showing the rotation of a vector in 45 degrees counterclockwise]

You are told that \( T \) is a linear transformation. (This will be shown in the next section.) Find the matrix of \( T \).

34. Consider the transformation \( T \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) that rotates any vector \( \vec{x} \) through a given angle \( \theta \) in the counterclockwise direction. Compare this with Exercise 33. You are told that \( T \) is linear. Find the matrix of \( T \) in terms of \( \theta \).
35. In the example about the French coast guard in this section, suppose you are a spy watching the boat and listening in on the radio messages from the boat. You collect the following data:

- When the actual position is $\begin{bmatrix} 5 \\ 42 \end{bmatrix}$, they radio $\begin{bmatrix} 89 \\ 52 \end{bmatrix}$.
- When the actual position is $\begin{bmatrix} 6 \\ 41 \end{bmatrix}$, they radio $\begin{bmatrix} 88 \\ 53 \end{bmatrix}$.

Can you crack their code (i.e., find the coding matrix), assuming that the code is linear?

36. Let $T$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}$ be three vectors in $\mathbb{R}^2$, as shown below. We are told that $T(\vec{v}_1) = \vec{v}_1$ and $T(\vec{v}_2) = 3\vec{v}_2$. On the same axes, sketch $T(\vec{v})$.

37. Consider a linear transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose that $\vec{v}$ and $\vec{w}$ are two arbitrary vectors in $\mathbb{R}^2$ and that $\vec{x}$ is a third vector whose endpoint is on the line segment connecting the endpoints of $\vec{v}$ and $\vec{w}$. Is the endpoint of the vector $T(\vec{x})$ necessarily on the line segment connecting the endpoints of $T(\vec{v})$ and $T(\vec{w})$? Justify your answer.

38. The two column vectors $\vec{v}_1$ and $\vec{v}_2$ of a $2 \times 2$ matrix $A$ are shown in the accompanying sketch. Consider the linear transformation $T(\vec{x}) = A\vec{x}$, from $\mathbb{R}^2$ to $\mathbb{R}^2$. Sketch the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

39. Show that if $T$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

$$T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_mT(\vec{e}_m),$$

where $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$ are the standard vectors in $\mathbb{R}^m$.

40. Describe all linear transformations from $\mathbb{R}$ (\(= \mathbb{R}^1\)) to $\mathbb{R}$. What do their graphs look like?

41. Describe all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}$ (\(= \mathbb{R}^1\)). What do their graphs look like?

42. When you represent a three-dimensional object graphically in the plane (on paper, the blackboard, or a computer screen), you have to transform spatial coordinates,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

into plane coordinates, $\begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}$. The simplest choice is a linear transformation, for example, the one given by the matrix

$$\begin{bmatrix} -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}.$$

a. Use this transformation to represent the unit cube with corner points

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Include the images of the $x_1$, $x_2$, and $x_3$ axes in your sketch:

b. Represent the image of the point $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ in your figure in part (a).

(part c on next page)
c. Find all the points \[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]
that are transformed to \[
\begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]. Explain.

43. a. Consider the vector \( \bar{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \). Is the transformation \( T(\bar{x}) = \bar{v} \cdot \bar{x} \) (the dot product) from \( \mathbb{R}^3 \) to \( \mathbb{R} \) linear? If so, find the matrix of \( T \).

b. Consider an arbitrary vector \( \bar{v} \) in \( \mathbb{R}^3 \). Is the transformation \( T(\bar{x}) = \bar{v} \cdot \bar{x} \) linear? If so, find the matrix of \( T \) (in terms of the components of \( \bar{v} \)).

c. Conversely, consider a linear transformation \( T \) from \( \mathbb{R}^3 \) to \( \mathbb{R} \). Show that there exists a vector \( \bar{v} \) in \( \mathbb{R}^3 \) such that \( T(\bar{x}) = \bar{v} \cdot \bar{x} \), for all \( \bar{x} \) in \( \mathbb{R}^3 \).

44. The cross product of two vectors in \( \mathbb{R}^3 \) is given by
\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix} \times \begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix} = \begin{bmatrix}
  a_2b_3 - a_3b_2 \\
  a_3b_1 - a_1b_3 \\
  a_1b_2 - a_2b_1
\end{bmatrix}.
\]
See Definition A.9 and Theorem A.11 in the Appendix. Consider an arbitrary vector \( \bar{v} \) in \( \mathbb{R}^3 \). Is the transformation \( T(\bar{x}) = \bar{v} \times \bar{x} \) from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) linear? If so, find its matrix in terms of the components of the vector \( \bar{v} \).

45. Consider two linear transformations \( \bar{v} = T(\bar{x}) \) and \( \bar{z} = L(\bar{y}) \), where \( T \) goes from \( \mathbb{R}^m \) to \( \mathbb{R}^p \) and \( L \) goes from \( \mathbb{R}^p \) to \( \mathbb{R}^n \). Is the transformation \( \bar{z} = L(T(\bar{x})) \) linear as well? [The transformation \( \bar{z} = L(T(\bar{x})) \) is called the \textit{composite} of \( T \) and \( L \).]

46. Let
\[
A = \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
  p & q \\
  r & s
\end{bmatrix}.
\]
Find the matrix of the linear transformation \( T(\bar{x}) = B(A\bar{x}) \). See Exercise 45. Hint: Find \( T(\bar{e}_1) \) and \( T(\bar{e}_2) \).

47. Let \( T \) be a linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). Three vectors \( \bar{v}_1, \bar{v}_2, \bar{w} \) in \( \mathbb{R}^2 \) and the vectors \( T(\bar{v}_1), T(\bar{v}_2) \) are shown in the accompanying figure. Sketch \( T(\bar{w}) \). Explain your answer.

48. Consider two linear transformations \( T \) and \( L \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). We are told that \( T(\bar{v}_1) = L(\bar{v}_1) \) and \( T(\bar{v}_2) = L(\bar{v}_2) \) for the vectors \( \bar{v}_1 \) and \( \bar{v}_2 \) sketched below. Show that \( T(\bar{x}) = L(\bar{x}) \), for all vectors \( \bar{x} \) in \( \mathbb{R}^2 \).

49. Prove that if \( A \) is a transition matrix and \( \bar{x} \) is a distribution vector, then \( A\bar{x} \) is a distribution vector as well.

For each of the mini-Webs in Exercises 50 through 52,

a. find the transition matrix \( A \) as defined in Example 9,

b. find the equilibrium distribution, and

c. find the web page(s) with the highest (naive) Page-Rank.

Feel free to use technology throughout.

50. 

\[
\begin{array}{c}
1 \\
4 \\
3 \\
\end{array} \quad \text{\( \rightarrow \)} \quad \begin{array}{c}
2 \\
1 \\
1 \\
\end{array} \quad \text{\( \uparrow \)} \quad \begin{array}{c}
3 \\
4 \\
\end{array}
\]

51. 

\[
\begin{array}{c}
1 \\
\uparrow\uparrow \\
3 \\
\end{array} \quad \text{\( \rightarrow \)} \quad \begin{array}{c}
2 \\
4 \\
\end{array}
\]

52. 

\[
\begin{array}{c}
1 \\
\uparrow \\
3 \\
\end{array} \quad \text{\( \rightarrow \)} \quad \begin{array}{c}
2 \\
4 \\
\end{array}
\]

53. As in Example 9, consider the mini-Web with the link structure shown in the diagram below:

\[
\begin{array}{c}
1 \\
\downarrow \uparrow \\
3 \\
\end{array} \quad \text{\( \rightarrow \)} \quad \begin{array}{c}
2 \\
4 \\
\end{array}
\]

Again, let vector \( \bar{x} \) represent the distribution of surfers among the various pages at a given time. Following the 1998 paper by Brin and Page (see Example 9), we consider a more sophisticated model of transition: At the blow of a whistle, 80% of the surfers on a given page will randomly follow a link, while the remaining 20% will randomly "jump" to a web page (even if no link is available). The jumpers have the option to stay on the current page. For example, of the surfers that are initially on page 2, 40% will follow the link to page 1, 40% will follow the link to page 3, and 5% will jump to any of the four webpages. If we add up the jumpers and the link-followers, we see that 45% will move to page 1; another 45% go to page 3; 5% will jump to page 4; and 5% will stay put on page 2. Thus, the second column of the transition matrix \( B \) will be
2.1 Introduction to Linear Transformations and Their Inverses

\[
\begin{bmatrix}
0.4 + 0.05 \\
0.05 \\
0.4 + 0.05 \\
0.05
\end{bmatrix} = 
\begin{bmatrix}
0.45 \\
0.05 \\
0.45 \\
0.05
\end{bmatrix}
\]

\[
\uparrow \quad \uparrow
\]

link jumpers

followers

Let \( \tilde{y} \) be the distribution vector after the transition; we have \( y_2 = 0.45x_1 + 0.05x_2 + 0.05x_3 + 0.85x_4 \), for example.

a. Find the transition matrix \( B \) such that \( \tilde{y} = B\tilde{x} \).

b. Explain why \( B = \frac{N}{E} E + 0.8A = 0.05E + 0.8A \), where \( A \) is the matrix we found in Example 9, \( E \) is the matrix whose entries are all 1, and \( N \) is the total number of web pages.

c. Find the equilibrium distribution \( \tilde{x}_{equ} \), such that \( B\tilde{x}_{equ} = \tilde{x}_{equ} \). Feel free to use technology. (The components of this equilibrium solution represent the PageRank of the various web pages, as defined in the 1998 paper by Brin and Page.)

For each of the mini-Webs in Exercises 54 through 56,

a. find the transition matrix \( B \) as defined in Exercise 53 (with a jumping rate of 20%),

b. find the equilibrium distribution for the transition matrix \( B \), and

c. find the web page(s) with the highest PageRank (with a jumping rate of 20%).

Feel free to use technology throughout.

54.

\[
\begin{align*}
1 & \rightarrow 2 \\
1 & \rightarrow \uparrow \\
3 & \rightarrow 4
\end{align*}
\]

55.

\[
\begin{align*}
1 & \rightarrow 2 \\
1 & \rightarrow \uparrow \\
1 & \rightarrow 4
\end{align*}
\]

56.

\[
\begin{align*}
1 & \rightarrow 2 \\
1 & \rightarrow \uparrow \\
3 & \rightarrow 4
\end{align*}
\]

57. Some parking meters in downtown Geneva, Switzerland, accept 2 Franc and 5 Franc coins.

a. A parking officer collects 51 coins worth 144 Francs. How many coins are there of each kind?

b. Find the matrix \( A \) that transforms the vector

\[
\begin{bmatrix}
\text{number of 2 Franc coins} \\
\text{number of 5 Franc coins}
\end{bmatrix}
\]

into the vector

\[
\begin{bmatrix}
\text{total value of coins} \\
\text{total number of coins}
\end{bmatrix}
\]

c. Is the matrix \( A \) in part (b) invertible? If so, find the inverse (use Exercise 13). Use the result to check your answer in part (a).

58. A goldsmith uses a platinum alloy and a silver alloy to make jewelry; the densities of these alloys are exactly 20 and 10 grams per cubic centimeter, respectively.

a. King Hiero of Syracuse orders a crown from this goldsmith, with a total mass of 5 kilograms (or 5,000 grams), with the stipulation that the platinum alloy must make up at least 90% of the mass. The goldsmith delivers a beautiful piece, but the king's friend Archimedes has doubts about its purity. While taking a bath, he comes up with a method to check the composition of the crown (famously shouting "Eureka!" in the process, and running to the king's palace naked). Submerging the crown in water, he finds its volume to be 370 cubic centimeters. How much of each alloy went into this piece (by mass)? Is this goldsmith a crook?

b. Find the matrix \( A \) that transforms the vector

\[
\begin{bmatrix}
\text{mass of platinum alloy} \\
\text{mass of silver alloy}
\end{bmatrix}
\]

into the vector

\[
\begin{bmatrix}
\text{total mass} \\
\text{total volume}
\end{bmatrix}
\]

for any piece of jewelry this goldsmith makes.

c. Is the matrix \( A \) in part (b) invertible? If so, find the inverse (use Exercise 13). Use the result to check your answer in part (a).

59. The conversion formula \( C = \frac{5}{9}(F - 32) \) from Fahrenheit to Celsius (as measures of temperature) is nonlinear, in the sense of linear algebra (why?). Still, there is a technique that allows us to use a matrix to represent this conversion.

a. Find the \( 2 \times 2 \) matrix \( A \) that transforms the vector

\[
\begin{bmatrix}
F \\
1
\end{bmatrix}
\]

into the vector \( \begin{bmatrix} C \end{bmatrix} \). (The second row of \( A \) will be \( \begin{bmatrix} 0 & 1 \end{bmatrix} \).)

b. Is the matrix \( A \) in part (a) invertible? If so, find the inverse (use Exercise 13). Use the result to write a formula expressing \( F \) in terms of \( C \).

60. In the financial pages of a newspaper, one can sometimes find a table (or matrix) listing the exchange rates between currencies. In this exercise we will consider a miniature version of such a table, involving only the Canadian dollar (CS) and the South African Rand (ZAR). Consider the matrix

\[
A = \begin{bmatrix}
1 & 1/8 \\
8 & 1
\end{bmatrix}
\]

CS ZAR

A = \[ \begin{bmatrix} 1 & 1/8 \end{bmatrix} \] CS

\[ \begin{bmatrix} 8 & 1 \end{bmatrix} \] ZAR
representing the fact that CS1 is worth ZAR8 (as of September 2012).

**a.** After a trip you have CS100 and ZAR1,600 in your pocket. We represent these two values in the vector \( \mathbf{x} = \begin{bmatrix} 100 \\ 1,600 \end{bmatrix} \). Compute \( A\mathbf{x} \). What is the practical significance of the two components of the vector \( A\mathbf{x} \)?

**b.** Verify that matrix \( A \) fails to be invertible. For which vectors \( \mathbf{b} \) is the system \( A\mathbf{x} = \mathbf{b} \) consistent? What is the practical significance of your answer? If the system \( A\mathbf{x} = \mathbf{b} \) is consistent, how many solutions \( \mathbf{x} \) are there? Again, what is the practical significance of the answer?

**61.** Consider a larger currency exchange matrix (see Exercise 60), involving four of the world’s leading currencies: Euro (€), U.S. dollar ($), Chinese yuan (¥), and British pound (£).

\[
A = \begin{bmatrix}
* & 0.8 & * & * \\
* & * & * & * \\
0.8 & * & * & * \\
* & * & 10 & * \\
\end{bmatrix}
\]

The entry \( a_{ij} \) gives the value of one unit of the \( i \)th currency, expressed in terms of the \( j \)th currency. For example, \( a_{34} = 10 \) means that €1 = ¥10 (as of August 2012). Find the exact values of the 13 missing entries of \( A \) (expressed as fractions).

**62.** Consider an arbitrary currency exchange matrix \( A \). See Exercises 60 and 61.

**a.** What are the diagonal entries \( a_{ii} \) of \( A \)?

**b.** What is the relationship between \( a_{ij} \) and \( a_{ji} \)?

**c.** What is the relationship among \( a_{ik}, a_{kj} \), and \( a_{ij} \)?

**d.** What is the rank of \( A \)? What is the relationship between \( A \) and \( \text{rref}(A) \)?

**63.** Solving a linear system \( A\mathbf{x} = \mathbf{b} \) by Gaussian elimination amounts to writing the vector of leading variables as a linear transformation of the vector of free variables. Consider the linear system

\[
\begin{align*}
2x_1 + 3x_2 + 4x_3 &= 0 \\
x_1 - x_2 + x_5 &= 0 \\
x_4 - 2x_5 &= 0.
\end{align*}
\]

Find the matrix \( B \) such that \( B\mathbf{x} = \mathbf{y} \).

**64.** Consider the linear system

\[
\begin{align*}
x_1 + 2x_2 + x_3 + 7x_4 &= 0 \\
x_1 + 2x_2 + x_3 + 11x_4 &= 0 \\
x_1 + 2x_2 + 3x_3 + 15x_4 &= 0 \\
x_1 + 2x_2 + 4x_3 + 19x_4 &= 0.
\end{align*}
\]

Find the matrix \( B \) such that \( B\mathbf{x} = \mathbf{y} \). See Exercise 63.

### 2.2 Linear Transformations in Geometry

In Example 2.1.5 we saw that the matrix \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) represents a counterclockwise rotation through 90° in the coordinate plane. Many other 2 × 2 matrices define simple geometrical transformations as well; this section is dedicated to a discussion of some of those transformations.

**EXAMPLE 1** Consider the matrices

\[
A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.
\]

Show the effect of each of these matrices on our standard letter L.\(^3\) and describe each transformation in words.

\(^3\)See Example 2.1.5. Recall that vector \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) is the foot of our standard L, and \( \begin{bmatrix} 0 \\ 2 \end{bmatrix} \) is its back.
\section*{Exercises 2.2}

**Goal**: Use the matrices of orthogonal projections, reflections, and rotations. Apply the definitions of shears, orthogonal projections, and reflections.

1. Sketch the image of the standard \( L \) under the linear transformation

\[
T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}.
\]

See Example 1.

2. Find the matrix of a rotation through an angle of 60° in the counterclockwise direction.

3. Consider a linear transformation \( T \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \). Use \( T(\vec{e}_1) \) and \( T(\vec{e}_2) \) to describe the image of the unit square geometrically.

4. Interpret the following linear transformation geometrically:

\[
T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}.
\]

5. The matrix

\[
\begin{bmatrix}
-0.8 & -0.6 \\
0.6 & -0.8
\end{bmatrix}
\]

represents a rotation. Find the angle of rotation (in radians).

6. Let \( L \) be the line in \( \mathbb{R}^3 \) that consists of all scalar multiples of the vector \( \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \). Find the orthogonal projection of the vector \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) onto \( L \).

7. Let \( L \) be the line in \( \mathbb{R}^3 \) that consists of all scalar multiples of \( \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \). Find the reflection of the vector \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) about the line \( L \).

8. Interpret the following linear transformation geometrically:

\[
T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \vec{x}.
\]

9. Interpret the following linear transformation geometrically:

\[
T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \vec{x}.
\]

10. Find the matrix of the orthogonal projection onto the line \( L \) in \( \mathbb{R}^2 \) shown in the accompanying figure:

11. Refer to Exercise 10. Find the matrix of the reflection about the line \( L \).

12. Consider a reflection matrix \( A \) and a vector \( \vec{x} \) in \( \mathbb{R}^2 \). We define \( \vec{v} = \vec{x} + A\vec{x} \) and \( \vec{w} = \vec{x} - A\vec{x} \).

   a. Using the definition of a reflection, express \( A(A\vec{x}) \) in terms of \( \vec{x} \).
   b. Express \( A\vec{v} \) in terms of \( \vec{v} \).
   c. Express \( A\vec{w} \) in terms of \( \vec{w} \).
   d. If the vectors \( \vec{v} \) and \( \vec{w} \) are both nonzero, what is the angle between \( \vec{v} \) and \( \vec{w} \)?
   e. If the vector \( \vec{v} \) is nonzero, what is the relationship between \( \vec{v} \) and the line \( L \) of reflection?

   Illustrate all parts of this exercise with a sketch showing \( \vec{x}, A\vec{x}, A(A\vec{x}), \vec{v}, \vec{w}, \) and the line \( L \).

13. Suppose a line \( L \) in \( \mathbb{R}^2 \) contains the unit vector \( \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \).

   Find the matrix \( A \) of the linear transformation \( T(\vec{x}) = \text{ref}_L(\vec{x}) \). Give the entries of \( A \) in terms of \( u_1 \) and \( u_2 \). Show that \( A \) is of the form \( \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \), where \( a^2 + b^2 = 1 \).

14. Suppose a line \( L \) in \( \mathbb{R}^3 \) contains the unit vector \( \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \).

   a. Find the matrix \( A \) of the linear transformation \( T(\vec{x}) = \text{proj}_L(\vec{x}) \). Give the entries of \( A \) in terms of the components \( u_1, u_2, u_3 \) of \( \vec{u} \).
   b. What is the sum of the diagonal entries of the matrix \( A \) you found in part (a)?

15. Suppose a line \( L \) in \( \mathbb{R}^3 \) contains the unit vector \( \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \).

   Find the matrix \( A \) of the linear transformation \( T(\vec{x}) = \text{ref}_L(\vec{x}) \). Give the entries of \( A \) in terms of the components \( u_1, u_2, u_3 \) of \( \vec{u} \).
16. Let \( T(\vec{x}) = \text{ref}_L(\vec{x}) \) be the reflection about the line \( L \) in \( \mathbb{R}^2 \) shown in the accompanying figure.
   a. Draw sketches to illustrate that \( T \) is linear.
   b. Find the matrix of \( T \) in terms of \( \theta \).

17. Consider a matrix \( A \) of the form \( A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \), where \( a^2 + b^2 = 1 \). Find two nonzero perpendicular vectors \( \vec{v} \) and \( \vec{w} \) such that \( A\vec{v} = \vec{v} \) and \( A\vec{w} = -\vec{w} \) (write the entries of \( \vec{v} \) and \( \vec{w} \) in terms of \( a \) and \( b \)). Conclude that \( T(\vec{x}) = A\vec{x} \) represents the reflection about the line \( L \) spanned by \( \vec{v} \).

18. The linear transformation \( T(\vec{x}) = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \vec{x} \) is a reflection about a line \( L \). See Exercise 17. Find the equation of line \( L \) (in the form \( y = mx \)).

Find the matrices of the linear transformations from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) given in Exercises 19 through 23. Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear.

19. The orthogonal projection onto the \( x-y \)-plane.
20. The reflection about the \( x-z \)-plane.
21. The rotation about the \( z \)-axis through an angle of \( \pi/2 \), counterclockwise as viewed from the positive \( z \)-axis.
22. The rotation about the \( y \)-axis through an angle \( \theta \), counterclockwise as viewed from the positive \( y \)-axis.
23. The reflection about the plane \( y = z \).

24. Rotations and reflections have two remarkable properties: They preserve the length of vectors and the angle between vectors. (Draw figures illustrating these properties.) We will show that, conversely, any linear transformation \( T \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) that preserves length and angles is either a rotation or a reflection (about a line).
   a. Show that if \( T(\vec{x}) = A\vec{x} \) preserves length and angles, then the two column vectors \( \vec{v} \) and \( \vec{w} \) of \( A \) must be perpendicular unit vectors.
   b. Write the first column vector of \( A \) as \( \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \), note that \( a^2 + b^2 = 1 \), since \( \vec{v} \) is a unit vector. Show that for a given \( \vec{v} \) there are two possibilities for \( \vec{w} \), the second column vector of \( A \). Draw a sketch showing \( \vec{v} \) and the two possible vectors \( \vec{w} \). Write the components of \( \vec{w} \) in terms of \( a \) and \( b \).
   c. Show that if a linear transformation \( T \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) preserves length and angles, then \( T \) is either a rotation or a reflection (about a line). See Exercise 17.

25. Find the inverse of the matrix \( \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \), where \( k \) is an arbitrary constant. Interpret your result geometrically.

26. a. Find the scaling matrix \( A \) that transforms \( \begin{bmatrix} 2 \\ -1 \end{bmatrix} \) into \( \begin{bmatrix} 8 \\ -4 \end{bmatrix} \).
   b. Find the orthogonal projection matrix \( B \) that transforms \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) into \( \begin{bmatrix} 2 \\ 0 \end{bmatrix} \).
   c. Find the rotation matrix \( C \) that transforms \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) into \( \begin{bmatrix} 3 \\ 4 \end{bmatrix} \).
   d. Find the shear matrix \( D \) that transforms \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) into \( \begin{bmatrix} 7 \\ 3 \end{bmatrix} \).
   e. Find the reflection matrix \( E \) that transforms \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) into \( \begin{bmatrix} -5 \\ 5 \end{bmatrix} \).

27. Consider the matrices \( A \) through \( E \) below.
\[
A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \\
C = \begin{bmatrix} 0.36 & -0.48 \\ -0.48 & 0.64 \end{bmatrix}, \quad D = \begin{bmatrix} -0.8 & -0.6 \\ -0.6 & -0.8 \end{bmatrix}, \\
E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
\]

Fill in the blanks in the sentences below.
We are told that there is a solution in each case.
Matrix ______ represents a scaling.
Matrix ______ represents an orthogonal projection.
Matrix ______ represents a shear.
Matrix ______ represents a reflection.
Matrix ______ represents a rotation.

28. Each of the linear transformations in parts (a) through (e) corresponds to one (and only one) of the matrices \( A \) through \( J \). Match them up.
a. Scaling b. Shear c. Rotation
d. Orthogonal projection e. Reflection
\[
A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix},
\[
D = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix},
\]

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\[
G = \begin{bmatrix}
0.6 & 0.6 \\
0.8 & 0.8
\end{bmatrix}, \quad H = \begin{bmatrix}
2 & -1 \\
1 & 2
\end{bmatrix}, \quad I = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}, \quad J = \begin{bmatrix}
0.8 & -0.6 \\
0.6 & -0.8
\end{bmatrix}
\]

29. Let \( T \) be a function from \( \mathbb{R}^m \) to \( \mathbb{R}^n \), and let \( L \) be a function from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). Suppose that \( L(T(\vec{x})) = \vec{x} \) for all \( \vec{x} \) in \( \mathbb{R}^m \) and \( T(L(\vec{y})) = \vec{y} \) for all \( \vec{y} \) in \( \mathbb{R}^n \). If \( T \) is a linear transformation, show that \( L \) is linear as well. \textit{Hint:} \( \vec{u} + \vec{v} = T(L(\vec{u})) + T(L(\vec{v})) = T(L(\vec{u}) + L(\vec{v})) \) since \( T \) is linear. Now apply \( L \) on both sides.

30. Find a nonzero \( 2 \times 2 \) matrix \( A \) such that \( A\vec{x} \) is parallel to the vector \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \), for all \( \vec{x} \) in \( \mathbb{R}^2 \).

31. Find a nonzero \( 3 \times 3 \) matrix \( A \) such that \( A\vec{x} \) is perpendicular to \( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \), for all \( \vec{x} \) in \( \mathbb{R}^3 \).

32. Consider the rotation matrix \( D = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \) and the vector \( \vec{v} = \begin{bmatrix} \cos \beta \\ \sin \beta \end{bmatrix} \), where \( \alpha \) and \( \beta \) are arbitrary angles.
   a. Draw a sketch to explain why \( D\vec{v} = \begin{bmatrix} \cos(\alpha + \beta) \\ \sin(\alpha + \beta) \end{bmatrix} \).
   b. Compute \( D\vec{v} \). Use the result to derive the addition theorems for sine and cosine:
     \[ \cos(\alpha + \beta) = \ldots \ldots \] \[ \sin(\alpha + \beta) = \ldots \ldots \]

33. Consider two nonparallel lines \( L_1 \) and \( L_2 \) in \( \mathbb{R}^2 \). Explain why a vector \( \vec{v} \) in \( \mathbb{R}^2 \) can be expressed uniquely as
   \[ \vec{v} = \vec{v}_1 + \vec{v}_2, \]
   where \( \vec{v}_1 \) is on \( L_1 \) and \( \vec{v}_2 \) on \( L_2 \). Draw a sketch. The transformation \( T(\vec{v}) = \vec{v}_1 \) is called the \textit{projection onto} \( L_1 \) \textit{along} \( L_2 \). Show algebraically that \( T \) is linear.

34. One of the five given matrices represents an orthogonal projection onto a line and another represents a reflection about a line. Identify both and briefly justify your choice.
   \[ A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad B = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad C = \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad D = -\frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad E = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \]

35. Let \( T \) be an invertible linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). Let \( P \) be a parallelogram in \( \mathbb{R}^2 \) with one vertex at the origin. Is the image of \( P \) a parallelogram as well? Explain. Draw a sketch of the image.

36. Let \( T \) be an invertible linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). Let \( P \) be a parallelogram in \( \mathbb{R}^2 \). Is the image of \( P \) a parallelogram as well? Explain.

37. The \textit{trace} of a matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is the sum \( a + d \) of its diagonal entries. What can you say about the trace of a \( 2 \times 2 \) matrix that represents \( \text{an} \) \textit{ orthogonal projection } \textit{b. reflection about a line} \textit{c. rotation } \textit{d. (horizontal or vertical) shear}. In three cases, give the exact value of the trace, and in one case, give an interval of possible values.

38. The \textit{determinant} of a matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is \( ad - bc \) (we have seen this quantity in Exercise 2.1.12 already). Find the determinant of a matrix that represents \( \text{an} \) \textit{ orthogonal projection } \textit{b. reflection about a line} \textit{c. rotation } \textit{d. (horizontal or vertical) shear}. What do your answers tell you about the invertibility of these matrices?

39. Describe each of the linear transformations defined by the matrices in parts (a) through (c) geometrically, as a well-known transformation combined with a scaling. Give the scaling factor in each case.
   \[ a. \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad b. \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix}, \quad c. \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \]

40. Let \( P \) and \( Q \) be two perpendicular lines in \( \mathbb{R}^2 \). For a vector \( \vec{x} \) in \( \mathbb{R}^2 \), what is \( \text{proj}_P(\vec{x}) + \text{proj}_Q(\vec{x}) \)? Give
your answer in terms of $\bar{x}$. Draw a sketch to justify your answer.

41. Let $P$ and $Q$ be two perpendicular lines in $\mathbb{R}^2$. For a vector $\bar{x}$ in $\mathbb{R}^2$, what is the relationship between $\text{ref}_P(\bar{x})$ and $\text{ref}_Q(\bar{x})$? Draw a sketch to justify your answer.

42. Let $T(\bar{x}) = \text{proj}_L(\bar{x})$ be the orthogonal projection onto a line in $\mathbb{R}^2$. What is the relationship between $T(\bar{x})$ and $T(T(\bar{x}))$? Justify your answer carefully.

43. Use the formula derived in Exercise 2.1.13 to find the inverse of the rotation matrix

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$ 

Interpret the linear transformation defined by $A^{-1}$ geometrically. Explain.

44. A nonzero matrix of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ represents a rotation combined with a scaling. Use the formula derived in Exercise 2.1.13 to find the inverse of $A$. Interpret the linear transformation defined by $A^{-1}$ geometrically. Explain.

45. A matrix of the form $A = \begin{bmatrix} a \\ b \\ -a \end{bmatrix}$, where $a^2 + b^2 = 1$, represents a reflection about a line. See Exercise 17. Use the formula derived in Exercise 2.1.13 to find the inverse of $A$. Explain.

46. A nonzero matrix of the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ represents a reflection about a line $L$ combined with a scaling. (Why? What is the scaling factor?) Use the formula derived in Exercise 2.1.13 to find the inverse of $A$. Interpret the linear transformation defined by $A^{-1}$ geometrically. Explain.

47. In this exercise we will prove the following remarkable theorem: If $T(\bar{x}) = A\bar{x}$ is any linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, then there exist perpendicular unit vectors $\bar{v}_1$ and $\bar{v}_2$ in $\mathbb{R}^2$ such that the vectors $T(\bar{v}_1)$ and $T(\bar{v}_2)$ are perpendicular as well (see the accompanying figure), in the sense that $T(\bar{v}_1) \cdot T(\bar{v}_2) = 0$. This is not intuitively obvious: Think about the case of a shear, for example. For a generalization, see Theorem 8.3.3.

For any real number $t$, the vectors

$$\begin{bmatrix} \cos t \\ -\sin t \\ \cos t \end{bmatrix}$$

will be perpendicular unit vectors. Now we can consider the function

$$f(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \cdot \begin{bmatrix} \cos t \\ -\sin t \\ \cos t \end{bmatrix} = A \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \cdot \begin{bmatrix} \cos t \\ -\sin t \\ \cos t \end{bmatrix}.$$ 

48. If a $2 \times 2$ matrix $A$ represents a rotation, find perpendicular unit vectors $\bar{v}_1$ and $\bar{v}_2$ in $\mathbb{R}^2$ such that the vectors $T(\bar{v}_1)$ and $T(\bar{v}_2)$ are perpendicular as well. See Exercise 47.

For the linear transformations $T$ in Exercises 49 through 52, do the following:

a. Find the function $f(t)$ defined in Exercise 47 and graph it for $0 \leq t \leq \frac{\pi}{2}$. You may use technology.

b. Find a number $c$, with $0 \leq c \leq \frac{\pi}{2}$, such that $f(c) = 0$. (In Problem 50, approximate $c$ to three significant digits, using technology.)

c. Find perpendicular unit vectors $\bar{v}_1$ and $\bar{v}_2$ in $\mathbb{R}^2$ such that the vectors $T(\bar{v}_1)$ and $T(\bar{v}_2)$ are perpendicular as well. Draw a sketch showing $\bar{v}_1$, $\bar{v}_2$, $T(\bar{v}_1)$, and $T(\bar{v}_2)$.

49. $T(\bar{x}) = \begin{bmatrix} 2 & 2 \\ 1 & -4 \end{bmatrix} \bar{x}$

50. $T(\bar{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \bar{x}$

51. $T(\bar{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \bar{x}$