Circles
Math 130 Kovits 2012

The graph of \( x^2 + y^2 = 1 \) is a unit circle with center \((0,0)\) and radius 1.

The graph of \( x^2 + y^2 = r^2 \) is a circle with center \((0,0)\) and radius \(r\).

Given any equation of the form

\[
x^2 + y^2 + Ax + By + C = 0,
\]

it can be put in the form

\[
(x - h)^2 + (y - k)^2 = s
\]

by completing the square.

For example:

\[
x^2 + y^2 + 8x - 12y + 43 = 0
\]

\[
x^2 + 8x + y^2 - 12y = -43
\]

\[
(x^2 + 8x + 16) + (y^2 - 12y + 36) = -43 + 16 + 36
\]

\[
(x + 4)^2 + (y - 6)^2 = 9
\]

This is a circle with radius 3, since \(r^2 = 9\) and \(x^2 + y^2 = 9\) is a circle with radius 3 and center \((0,0)\). As we will later learn, the substitutions of \(x + 4\) for \(x\) and \(y - 6\) for \(y\) cause shifts of 4 to the left and 6 up. The center, originally at \((0,0)\) is shifted 4 to the left and 6 up, landing at the point \((-4,6)\).

Be Careful. Add the completing numbers for \(x\) and \(y\) to both sides of the equation and remember that \(h\) and \(k\) in \((h,k)\) have signs opposite to the signs separating \(x\) and \(y\) from the numbers in the completed square version of the equation. The equation in the example above could have been rewritten as \((x - (-4))^2 + (y - 6)^2 = 9\), with \(h = -4\) and \(k = 6\).
CIRCLES

General Form of the Equation of a Circle:

When \( s \) is positive, the graph of

\[
(x - h)^2 + (y - k)^2 = s
\]

is a circle with center \((h, k)\) and radius \(\sqrt{s}\). Two questions remain:

1) What if \((x - h)^2 + (y - k)^2 = s\) and \(s\) is negative?

No solution is possible since the left side is positive or zero and the right side is negative. There is no graph—the solution set is empty.

2) What if \((x - h)^2 + (y - k)^2 = 0\)?

The sum of two expressions, each a square and therefore greater than or equal to zero, is equal to zero. The only possibility is for each expression to equal zero. The solution set is a single point, \((h, k)\).

For example:

\[
(x - 3)^2 + (y + 2)^2 = 0
\]

The solution set is the point \((3, -2)\) only. This case is called a degenerate circle.

The case

\[
Dx^2 + Dy^2 + Ax + By + C = 0
\]

has, of course, a graph which is also a circle, a single point, or no points. You just divide the expression by \(D\) as the first step.

For example:

\[
3x^2 + 3y^2 + 24x - 36y + 129 = 0
\]

is the same graph as the example on page 1. First divide by 3 to obtain the same equation as in the first example.

The square must be completed by taking the coefficient of the \(x\) term, dividing by 2 and then squaring. The same procedure is then used for the \(y\) term. The terms involving \(x^2\) and \(y^2\) should have coefficients equal to 1.

The procedure was

\[
\begin{align*}
8x & \quad \to \quad 8 \quad \to \quad 4 \quad \to \quad 16 \\
-12y & \quad \to \quad -12 \quad \to \quad -6 \quad \to \quad 36.
\end{align*}
\]

Alternate Method:

From \(x^2 + y^2 + Ax + By + C = 0\) we get \(h = -A/2\) and \(k = -B/2\). Remember to put the \(-\) in front of the \(h\) and \(k\) when they are placed into the equation.

The example redone in this alternate method would be as follows.

\[
\begin{align*}
\text{(for } x) \quad h &= -8/2 = -4 \\
\text{(for } y) \quad k &= -(12)/2 = 12/2 = 6
\end{align*}
\]

\[
(x + 4)^2 + (y - 6)^2 = -43 + (4)^2 + (-6)^2
\]

The completing numbers, \((4)^2\) and \((-6)^2\), were added to both sides of the equation.