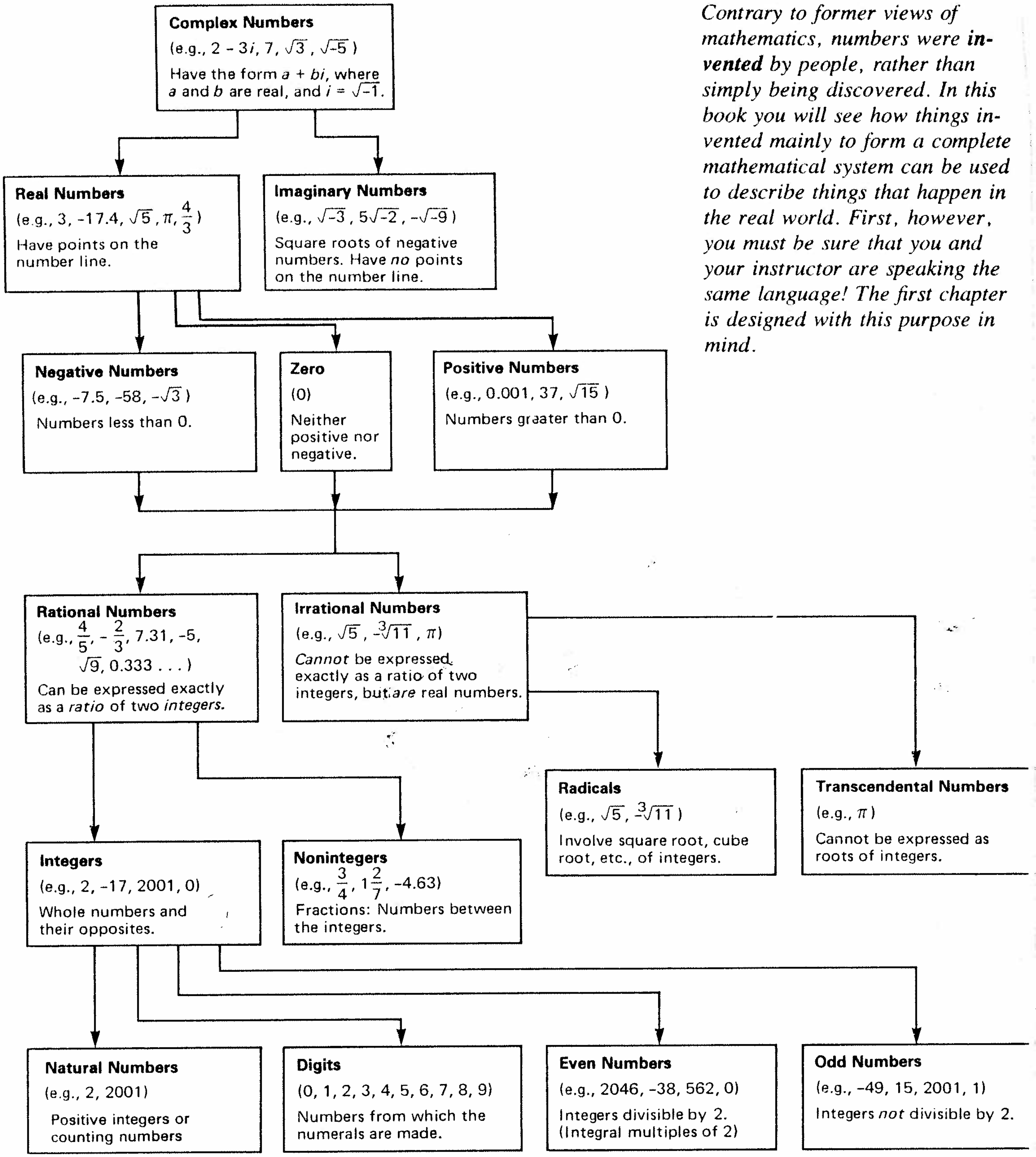
1 Preliminary Information

*Contrary to former views of mathematics, numbers were **invented** by people, rather than simply being discovered. In this book you will see how things invented mainly to form a complete mathematical system can be used to describe things that happen in the real world. First, however, you must be sure that you and your instructor are speaking the same language! The first chapter is designed with this purpose in mind.*



1-1 SETS OF NUMBERS

From previous work in mathematics you should recall the names of different kinds of numbers (positive, even, irrational, etc.). In this section you will refresh your memory so that you will know the exact meaning of these names.

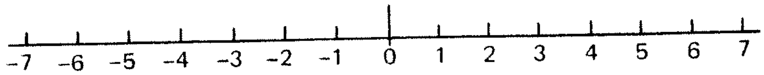
Objective:

Given the name of a set of numbers, provide an example; or given a number, name the sets to which it belongs.

There are two major sets of numbers you will deal with in this course, the *real* numbers and the *imaginary* numbers. The real numbers are given this name because they are used for "real" things such as measuring and counting. The imaginary numbers are square roots of negative numbers. They are useful, too, but you must learn more mathematics to see why.

The real numbers are all numbers which you can plot on a number line (see Figure 1-1). They can be broken into subsets in several ways. For instance, there are positive and negative real numbers, integers and non-integers, rational and irrational real numbers, and so forth. The diagram facing this page shows some subsets of the set of real numbers.

The numbers in the diagram were invented in *reverse* order. The natural (or "counting") numbers came first because mathematics was first used for counting. The negative numbers (those less than zero) were invented so that there would always be answers to subtraction problems. The rational numbers were invented to provide answers to division problems, and the irrational ones came when it was shown that numbers such as $\sqrt{2}$ could not be expressed as a ratio of two integers.



The real number line

Figure 1-1

Other operations you will invent, such as taking logarithms and cosines, lead to irrational numbers which go beyond even extracting roots. These are called "transcendental" numbers, meaning "going beyond." When all of these various kinds of numbers are put together, you get the set of *real* numbers. The imaginary numbers were invented because no real number squared equals a negative number. Later, you will see that the real and imaginary numbers are themselves simply subsets of a larger set, called the "complex numbers."

The following exercise is designed to help you accomplish the objectives of this section.

EXERCISE 1-1

1. Write a definition for each of the following sets of numbers. Try to do this *without* referring to the diagram opposite page 1. Then look to make sure you are correct.
   a. {integers}
   b. {digits}
   c. {even numbers}
   d. {positive numbers}
   e. {negative numbers}
   f. {rational numbers}
   g. {irrational numbers}
   h. {imaginary numbers}
   i. {real numbers}
   j. {natural numbers}
   k. {counting numbers}
   l. {transcendental nos.}
2. Write an example of each type of number mentioned in Problem 1.
3. Copy the chart at right. Put a check mark in each box for which the number on the left of the chart belongs to the set across the top.
4. Write another name for {natural numbers}.
5. Which of the sets of numbers in Problem 1 do you suppose was the *first* to be invented? Why?
6. One of the sets of numbers in Problem 1 contains all but one of the others as subsets.
   a. Which one *contains* the others?
   b. Which one is left out?
7. Do decimals such as 2.718 represent *rational* numbers or *irrational* numbers? Explain.
8. Do repeating decimals such as 2.3333 . . . represent *rational* numbers or *irrational* numbers? Explain.
9. What real number is neither positive nor negative?

	Integers	Digits	Even Numbers	Positive Numbers	Negative Numbers	Rational Numbers	Irrational Numbers	Imaginary Numbers	Real Numbers	Natural Numbers	Counting Numbers	Transcendental Numbers
a. 5												
b. 2/3												
c. -7												
d. $\sqrt{3}$												
e. $\sqrt{16}$												
f. $\sqrt{-16}$												
g. $\sqrt{-15}$												
h. 44												
i. π												
j. 1.765												
k. -10000												
l. $-1\frac{1}{2}$												
m. $-\sqrt{6}$												
n. 0												
o. 1												
p. 1/9												

1-2 THE FIELD AXIOMS

From previous mathematics courses you probably remember names such as "Distributive Property," "Reflexive Property," and "Multiplication Property of Zero." Some of these properties, called *axioms*, are accepted without proof and are used as starting points for working with numbers.

From a small number of rather obvious axioms, you will derive all the other properties you will need. In this section you will concentrate on the axioms that apply to the *operations* with numbers such as + and ×. In Section 1-7 you will find the axioms that apply to the *relationships* between numbers, such as = and <.

Objective:

Given the name of an axiom that applies to + or ×, give an example that shows you understand the meaning of the axiom; and vice versa.

There are eleven axioms that apply to adding and multiplying real numbers. These are called the *Field Axioms*, and are listed in the following table. If you already feel familiar with these axioms, you may go right to the problems in Exercise 1-2. If not, then read on!

THE FIELD AXIOMS

CLOSURE

{real numbers} is *closed* under addition and under multiplication. That is, if x and y are real numbers, then

$x + y$ is a ***unique, real*** number,
xy is a ***unique, real*** number.

COMMUTATIVITY

Addition and multiplication of real numbers are *commutative* operations. That is, if x and y are real numbers, then

$x + y$ and $y + x$ are *equal* to each other,
xy and yx are equal to each other.

ASSOCIATIVITY

Addition and multiplication of real numbers are *associative* operations. That is, if x, y, and z are real numbers, then

$(x + y) + z$ and $x + (y + z)$ are *equal* to each other,
$(xy)z$ and $x(yz)$ are *equal* to each other.

DISTRIBUTIVITY

Multiplication *distributes* over addition. That is, if x, y, and z are real numbers, then

$x(y + z)$ and $xy + xz$ are *equal* to each other.

IDENTITY ELEMENTS

{real numbers} contains:

A ***unique*** identity element for ***addition***, namely 0. (Because $x + 0 = x$ for any real number x.)

A ***unique*** identity element for ***multiplication***, namely 1. (Because $x \cdot 1 = x$ for any real number x.)

INVERSES

{real numbers} contains:

A *unique additive* inverse for every real number x. (Meaning that every real number x has a real number $-x$ such that $x + (-x) = 0$.)

A *unique multiplicative* inverse for every real number x except zero.
(Meaning that every non-zero number x has a real number $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = 1$.)

Notes:

1. Any set that obeys all eleven of these axioms is a *field*.
2. The eleven Field Axioms come in 5 pairs, one of each pair being for addition and the other for multiplication. The Distributive Axiom expresses a relationship between these two operations.
3. The properties $x + 0 = x$ and $x \cdot 1 = x$ are sometimes called the "Addition Property of 0" and the "Multiplication Property of 1," respectively, for obvious reasons.
4. The number $-x$ is called, "the *opposite* of x," "the *additive inverse* of x," or "negative x."
5. The number $\frac{1}{x}$ is called the "multiplicative inverse of x," or the "reciprocal of x."

Closure—By saying that a set is "closed" under an operation, you mean that you cannot get an answer that is *out* of the set by performing that operation on numbers *in* the set. For example, {0, 1} is closed under multiplication because $0 \times 0 = 0$, $0 \times 1 = 0$, $1 \times 0 = 0$, and $1 \times 1 = 1$. All the answers are *unique*, and are *in* the given set. This set is *not* closed under addition because $1 + 1 = 2$, and 2 is *not* in the set. It is not closed under the operation "taking the square root" since there are *two* different square roots of 1: $+1$ and -1.

Commutativity—The word "commute" comes from the Latin word "commutare," which means "to exchange." People who travel back and forth between home and work are called "commuters" because they regularly exchange positions. The fact that addition and multiplication are commutative operations is somewhat unusual. Many operations such as subtraction and exponentiation (raising to powers) are *not* commutative. For example,

$$2 - 5 \text{ does } \textit{not} \text{ equal } 5 - 2,$$

and

$$2^3 \text{ does } \textit{not} \text{ equal } 3^2.$$

Indeed, most operations in the real world are not commutative. Putting on your shoes and socks (in that order) produces a far different result from putting on your socks and shoes!

Associativity—You can remember what this axiom states by remembering that to "associate" means to "group." Addition and multiplication are associative, as shown by

$$(2 + 3) + 4 = 9 \quad \text{and} \quad 2 + (3 + 4) = 9.$$

But subtraction is *not* associative. For example,

$$(2 - 3) - 4 = -5 \quad \text{and} \quad 2 - (3 - 4) = 3.$$

Distributivity—Parentheses in an expression such as $2 \times (3 + 4)$ mean, "Do what is inside *first*." But you don't *have* to do $3 + 4$ first. You could "distribute" a 2 to each term inside the parentheses, getting $2 \times 3 + 2 \times 4$. The Distributive Axiom expresses the fact that you get the same answer either way. That is,

$$2 \times (3 + 4) = 14 \quad \text{and} \quad 2 \times 3 + 2 \times 4 = 14.$$

Note that multiplication does *not* distribute over multiplication. For example,

$$2 \times (3 \times 4) \quad \text{does } \textit{not} \text{ equal} \quad 2 \times 3 \times 2 \times 4,$$

as you can easily check by doing the arithmetic.

Identity Elements—The numbers 0 and 1 are called "identity elements" for adding and multiplying, respectively, since a number comes out "identical" if you add 0 or multiply by 1. For example,

$$5 + 0 = 5 \quad \text{and} \quad 5 \times 1 = 5.$$

Inverses—A number is said to be an *inverse* of another number for a certain operation if it "undoes" (or inverts) what the other number did. For example, $\frac{1}{3}$ is the multiplicative inverse of 3. If you start with 5 and multiply by 3 you get

$$5 \times 3 = 15.$$

Multiplying the answer, 15, by $\frac{1}{3}$ gives

$$15 \times \frac{1}{3} = 5,$$

which "undoes" or "inverts" the multiplication by 3. It is easy to tell if two numbers are *multiplicative inverses* of each other because their product is always equal to 1, the multiplicative identity element. For example,

$$3 \times \frac{1}{3} = 1.$$

Similarly, two numbers are *additive inverses* of each other if adding them to each other gives 0, the additive identity element. For example, $\frac{5}{7}$ and $-\frac{5}{7}$ are additive inverses of each other because

$$\tfrac{5}{7} + (-\tfrac{5}{7}) = 0.$$

The following exercise is designed to familiarize you with the names and meanings of the Field Axioms.

EXERCISE 1-2

Do These Quickly

The following problems are intended to refresh your skills. Some problems come from the last section, and others probe your general knowledge of mathematics. You should be able to do all 10 in less than 5 minutes.

Q1. Simplify: $11 - 3 + 5$

Q2. Multiply and simplify: $\left(\frac{2}{3}\right)\left(\frac{6}{7}\right)$

Q3. Add: $3.74 + 5$

Q4. If $x + 7$ is 42, what does x equal?

Q5. Is -13 an integer?

Q6. Multiply: $(9x)(6x)$

Q7. Square 7.

Q8. Is 1.3 a rational number?

Q9. Multiply: $5(3x - 8)$

Q10. Simplify: $(-3)(0.7)(-5)(-1)$

Work the following problems.

1. Tell what is meant by
   a. additive identity element,
   b. multiplicative identity element.
2. What is
   a. the *additive* inverse of $\frac{2}{3}$?
   b. the *multiplicative* inverse of $\frac{2}{3}$?
3. Using variables (x, y, z, etc.) to stand for numbers, write an example of each of the eleven field axioms. Try to do this by writing all eleven

examples first, then checking to be sure you are right. Correct any which you left out or got wrong.

4. Explain why 0 has *no* multiplicative inverse.

5. The Closure Axiom states that you get a *unique* answer when you add two real numbers. What is meant by a "unique" answer?

6. You get the same answer when you add a column of numbers "up" as you do when you add it "down." What axiom(s) show that this is true?

7. Calvin Butterball and Phoebe Small use the distributive property as follows:

   Calvin: $3(x + 4)(x + 7) = (3x + 12)(x + 7)$.
   Phoebe: $3(x + 4)(x + 7) = (3x + 12)(3x + 21)$.

   Who is right? What mistake did the other one make?

8. Write an example which shows that:
   a. Subtraction is *not* a commutative operation.
   b. {negative numbers} is *not* closed under multiplication.
   c. {digits} is *not* closed under addition.
   d. {real numbers} is *not* closed under the $\sqrt{\ }$ operation (taking the square root).
   e. Exponentiation ("raising to powers") is *not* an associative operation. (Try 4^{2^3}.)

9. For each of the following, tell which of the Field Axioms was used, and whether it was an axiom for *addition* or for *multiplication*. Assume that x, y, and z stand for real numbers.
   a. $x + (y + z) = (x + y) + z$
   b. $x \cdot (y + z)$ is a real number
   c. $x \cdot (y + z) = x \cdot (z + y)$
   d. $x \cdot (y + z) = (y + z) \cdot x$
   e. $x \cdot (y + z) = xy + xz$
   f. $x \cdot (y + z) = x \cdot (y + z) + 0$
   g. $x \cdot (y + z) + (-[x \cdot (y + z)]) = 0$
   h. $x \cdot (y + z) = x \cdot (y + z) \cdot 1$
   i. $x \cdot (y + z) \cdot \dfrac{1}{x \cdot (y + z)} = 1$

10. Tell whether or not the following sets are *fields* under the operations $+$ and $\times$. If the set is not a field, tell which one(s) of the Field Axioms do not apply.
    a. {rational numbers}
    b. {integers}
    c. {positive numbers}
    d. {non-negative numbers}

1-3 VARIABLES AND EXPRESSIONS

In previous mathematics courses you have seen *expressions*, such as

$$3x^2 + 5x - 7,$$

that stand for numbers. Just *what* number an expression stands for depends on what value you pick for the *variable* (x in this case). The name "variable" is picked because x can stand for various different numbers at different times. The numbers 3, 5, -7, $\frac{9}{10}$, $\sqrt{11}$ etc., are called *constants* because they stand for the *same* number *all* the time.

In this section you will *evaluate* expressions by substituting values for the variable. In order to do this more easily, you can *simplify* the expression using the axioms of the previous section.

Objective:

Given an expression containing a variable,

a. *evaluate* it by substituting a given number for the variable, and finding the value of the expression,

b. *simplify* it by using the Field Axioms to transform it to an equivalent expression that is easier to evaluate.

DEFINITION

> **VARIABLE**
> A **variable** is a letter which stands for an *unspecified* number from a *given* set.

For example, if the set you have in mind is {digits}, and x is the variable, then x could stand for any one of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. In this case, {digits} is called the *domain* of x. The word comes from the Latin "domus," meaning "house." So the domain of a variable is "where it lives." Since the domain of most variables in this course will be {real numbers}, you make the following agreement:

AGREEMENT

> Unless otherwise specified, the domain of a variable will be assumed to be the set of all real numbers.

DEFINITION

EXPRESSION
An **expression** is a collection of variables and constants connected by operation signs ($+$, $-$, $\times$, $\div$, etc.) which stands for a *number*.

To find out *what* number an expression stands for, you must substitute a value for each variable, then do the indicated operations.

EXAMPLE 1

Evaluate $3x^2 + 5x - 7$ if $x = 4$.

Solution:

$3x^2 + 5x - 7$	Write the given expression.
$= 3 \cdot 4^2 + 5 \cdot 4 - 7$	Substitute 4 for x.
$= 3 \cdot 16 + 5 \cdot 4 - 7$	Square the 4.
$= 48 + 20 - 7$	Do the multiplication.
$= \underline{61}$	Add and subtract from left to right.

There are several things you should realize about the preceding calculations. First, you must substitute the *same* value of x *everywhere* it appears in the expression. Although a variable can take on different values at different times, it stands for the *same* number at any *one* time. This fact is expressed in the Reflexive Axiom, which states, "$x = x$."

The second thing you should realize is that this expression involves *subtraction* and *exponentiation* (raising to powers). These operations, as well as *division,* can be defined in terms of addition and multiplication.

DEFINITIONS

Subtraction: $x - y$ means $x + (-y)$.

Division: $x \div y$ means $x \cdot \frac{1}{y}$. (The symbols $\frac{x}{y}$ and x/y are also used for $x \div y$.)

Exponentiation: x^n means n, x's *multiplied* together. For example,

$$x^3 \text{ means } x \cdot x \cdot x.$$

The third thing you should realize is that the answer you get depends on the *order* in which you do the operations. So that there will be no doubt about what an expression such as $3x^2 + 5x - 7$ means, you make the following agreement:

AGREEMENT

ORDER OF OPERATIONS

1. Do any operations inside parentheses *first*.
2. Do any exponentiating next.
3. Do multiplication and division in the order in which they occur, from left to right.
4. Do addition and subtraction last, in the order in which they occur, from left to right.

EXAMPLE 2

Carry out the following operations:

a. $3 + 4 \times 5$ — Multiply *first*.

$= 3 + 20$

$= \underline{\underline{23}}$ — Add *last*.

b. $3 + 4 \times 5 \div 2$

$= 3 + 20 \div 2$ — Multiply and divide from left to right.

$= 3 + 10$ — Divide *before* adding.

$= \underline{\underline{13}}$ — Add *last*.

c. $3 - 4 \times 5 \div 2 + 9$

$= 3 - 20 \div 2 + 9$ — Multiply and divide from left to right.

$= 3 - 10 + 9$ — Divide *before* + and −.

$= -7 + 9$ — Add and subtract from left to right.

$= \underline{\underline{2}}$ — Add and subtract last.

An expression might contain the *absolute value* operation. The symbol $|x|$ means the *distance* between the number x and the origin of the number line. For example, $|-3|$ and $|3|$ are both equal to 3, since both 3 and -3 are located 3 units from the origin (Figure 1-3).

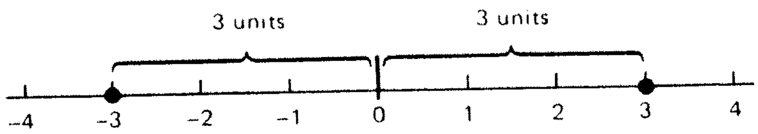


Figure 1-3

Similarly,

$$|5| = 5$$
$$|-7| = 7$$
$$|0| = 0,$$

and so forth.

The absolute value of a *variable* presents a problem. If x is a *positive* number, then $|x|$ is equal to x. But if x is a *negative* number, then $|x|$ is equal to the *opposite* of x. For instance, if $x = -9$, then

$$|x| = |-9| = -(-9) = 9.$$

A precise definition of absolute value can be written as follows:

DEFINITION

$|x| = x$ if x is *positive* (or 0)

$|x| = -x$ if x is *negative*

EXAMPLE 3

Evaluate $|17 - 4x| - 2$ if

a. $x = 5$
b. $x = -3$.

a. $|17 - 4x| - 2$

$= |17 - 20| - 2$ Substitute 5 for x.

$= |-3| - 2$ Arithmetic

$= 3 - 2$ Definition of absolute value

$= \underline{\underline{1}}$ Arithmetic

b. $|17 - 4x| - 2$

$= |17 + 12| - 2$ Substitute -3 for x.

$= |29| - 2$ Arithmetic

$= 29 - 2$ Definition of absolute value

$= \underline{\underline{27}}$ Arithmetic ∎

Two expressions are equivalent if they *equal* each other for *all* values of the variable. For example, $3x + 8x$ and $11x$ are equivalent expressions. *Simplifying* an expression means transforming it to an equivalent expression that is in some way simpler to work with. The expression $11x$ is considered to be simpler than $3x + 8x$ because it is easier to evaluate when you pick a value of x. Adding the $3x$ and $8x$ is called "collecting like terms." It is justified by using the Distributive Axiom *backwards*.

$3x + 8x = (3 + 8)x$ Distributivity

$= 11x$ Arithmetic

EXAMPLE 4

Simplify $7x \cdot 2 \div x$.

Since the Field Axioms apply to multiplication rather than division, you would treat "$\div x$" as "$\cdot \frac{1}{x}$", commute the multiplication, and get

$7x \cdot 2 \div x$

$= 7x \cdot 2 \cdot \frac{1}{x}$ Definition of division

$= \left(7x \cdot \frac{1}{x}\right) \cdot 2$ Commutativity and associativity

$= 7 \cdot 2$ Associativity and multiplicative inverses

$= \underline{\underline{14}}$ Arithmetic ∎

EXAMPLE 5

Simplify $2 - 3[x - 2 - 5(x - 1)]$.

Here you must observe the agreed-upon sequence of operations. The first thing to do is *start inside the innermost parentheses* and work your way out (like a termite!).

$2 - 3[x - 2 - 5(x - 1)]$

$= 2 - 3[x - 2 - 5x + 5]$ Distributivity

$= 2 - 3[-4x + 3]$ Collecting like terms

$= 2 + 12x - 9$ Distributivity

$= \underline{\underline{12x - 7}}$ Commutativity and associativity ∎

Notes:

1. You must remember some things from previous mathematics courses. For example, a negative number times a negative number is a *positive* number. This sort of thing can be proved using the Field Axioms, as you will see in Section 1-7.
2. There are several kinds of symbols of inclusion.
   ( ) Parentheses.
   [ ] Brackets.
   { } Braces (also used for *set* symbols).
   $\overline{\quad}$ Vinculum (an overhead line, used in fractions and elsewhere, such as in $\dfrac{x-3}{x+7}$).

   To avoid so many different symbols, sometimes "nested" parentheses are used. For example, the expression

   $$2 - (3 + 4(5 - 6(7 + x)))$$

   would be simplified by starting with the *innermost* parentheses.

In the following exercise, you will practice simplifying and evaluating expressions. If the going gets difficult, just tell yourself that no matter how complicated an expression looks, it just stands for a *number*. And people *invented* numbers!

EXERCISE 1-3

Do These Quickly

The following problems are intended to refresh your skills. Some are from the first two sections of this chapter, and others probe your general knowledge of mathematics. You should be able to do all 10 in less than 5 minutes.

Q1. Is $\sqrt{9}$ an integer?

Q2. Is $-\frac{4}{7}$ a real number?

Q3. Commute the 3 and the x: $2y + 3 + x$

Q4. Associate the $4a$ and the $2c$: $4a + 2c + 5d$

Q5. Distribute the 5: $5(3x - 7)$

Q6. Write the additive inverse of $\frac{5}{8}$.

Q7. Write the multiplicative identity element.

Q8. If $3x$ equals 42, what does x equal?

Q9. Multiply: (2.3)(4)

Q10. Divide and simplify: $(\frac{2}{3}) \div (\frac{6}{7})$

For Problems 1 through 10, carry out the indicated operations in the agreed-upon order.

1. $5 + 6 \times 7$
2. $3 + 8 \times 7$
3. $9 - 4 + 5$
4. $11 - 6 + 4$
5. $12 \div 3 \times 2$
6. $18 \div 9 \times 2$
7. $7 - 8 \div 2 + 4$
8. $24 - 12 \times 2 + 4$
9. $16 - 4 + 12 \div 6 \times 2$
10. $50 - 30 \times 2 + 8 \div 2$

For Problems 11 through 24, evaluate the given expression

(a) for $x = 2$
(b) for $x = -3$.

11. $4x - 1$
12. $3x - 5$
13. $|3x - 5|$
14. $|4x - 1|$
15. $5 - 7x - 8$
16. $8 - 5x - 2$
17. $|8 - 5x| - 2$
18. $|5 - 7x| - 8$
19. $x^2 - 4x + 6$
20. $x^2 + 6x - 9$
21. $4x^2 - 5x - 11$
22. $5x^2 - 7x + 1$
23. $5 - 2 \cdot x$
24. $3 + 4 \cdot x$

For Problems 25 through 40, simplify the given expression.

25. $6 - [5 - (3 - x)]$
26. $2x - [3x + (x - 2)]$
27. $7(x - 2(3 - x))$
28. $3(6x - 5(x - 1))$
29. $7 - 2[3 - 2(x + 4)]$
30. $8 + 4[5 - 6(x - 2)]$
31. $3x - [2x + (x - 5)]$
32. $4x - [3x - (2x - x)]$
33. $6 - 2[x - 3 - (x + 4) + 3(x - 2)]$
34. $7[2 - 3(x - 4) + 4(x - 6)]$
35. $6[x - \frac{1}{2}(x - 1)]$
36. $8[2x - \frac{1}{4}(6x + 5)]$

37. $x^2 + y^2 - [x(x + y) - y(y - x)]$

38. $4x^2 - 2x(x - 2y) + 2y(2y + x) - 2x^2$

39. $-(-(-(-x)))$

40. $x - [x - (x - \overline{x - y})]$

41. Calvin Butterball and Phoebe Small evaluate the expression $|x - 3|$ for $x = 7$, getting:

$$\text{Calvin: } |x - 3| = |7 - 3| = 7 + 3 = \underline{\underline{10}}$$

$$\text{Phoebe: } |x - 3| = |7 - 3| = |4| = \underline{\underline{4}}$$

Who is right? What mistake did the other one make?

42. Kay Oss evaluates the expression $|x + 2| - 5x$ by substituting 7 for the first x and 3 for the second x. What axiom did Kay violate?

1-4 POLYNOMIALS

Polynomials are algebraic expressions that involve only the operations of *addition, subtraction,* and *multiplication* of variables. For example,

$$3x^2 + 5x - 7, \quad x + 2, \quad \text{and} \quad xy^3z^2$$

are polynomials. They involve no non-algebraic operations such as absolute value, and no operations under which the set of real numbers is not closed, such as division and square root. Thus, polynomials stand for *real* numbers no matter what real values you substitute for the variables.

Objectives:

1. Given an expression, tell whether or not it is a polynomial. If it is, then *name* it by "degree" and by number of terms.
2. Given two binomials, multiply them together.

Notes:

1. The expression $\frac{3}{x-5}$ is *not* a polynomial since it involves *division* by a *variable*. If x were 5, the expression would have the form $\frac{3}{0}$, which is *not* a real number.
2. The expression $\sqrt{x}$ is *not* a polynomial since it involves the *square root* of a *variable*. If x were less than 0, the expression would stand for an imaginary number rather than a real number.

3. The expression $|x - 7|$ is *not* a polynomial since it involves the *non-algebraic operation* "absolute value."
4. Expressions such as $\sqrt{3}x$ and $\frac{x}{3}$ (which equals $\frac{1}{3} \cdot x$) *are* considered to be polynomials since the operations $\div$ and $\sqrt{\ }$ are performed on *constants* rather than variables.
5. The operation exponentiation ("raising to powers") is *not* listed among the polynomial operations. If the exponent is an integer, such as in x^4, then exponentiation is just repeated multiplication. So expressions with only *integer* exponents *are* polynomials. In Chapter 6 you will learn what happens when the exponent is not an integer.

"Terms" in an expression are parts of the expression that are *added* or *subtracted*. For example, the expression

$$3x^2 + 5x - 7$$

has three terms, namely, $3x^2$, $5x$, and 7. Special names are used for expressions that have 1, 2, or 3 terms.

NAMES

No. of Terms	Name	Example
1	monomial	$3x^2y^5$
2	binomial	$3x^2 + y^5$
3	trinomial	$3 - x^2 + y^5$
4 or more	(no special name)	$3x^5 - 2x^4 + 5x^3 - 6x^2 + 2x$

The word "polynomial" originally meant "many terms." However, it is possible to get a *monomial* by adding two polynomials. For example,

$$(3x^2 + 5x - 7) + (8x^2 - 5x + 7) = 11x^2,$$

a *monomial*. By calling monomials, binomials, and trinomials "polynomials," too, the set of polynomials has the desirable property of being *closed* under addition. It is also closed under multiplication.

"Factors" in an expression are parts of the expression that are *multiplied* together. For example, $5x^2$ has *three* factors, 5, x, and x. Special names are given to polynomials depending on how many *variables* are *multiplied* together.

For example, $3x^2y^5$ is *seventh* degree because seven variables are multiplied together $(x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y)$. But $3x^2 + y^5$ is only *fifth* degree because at most five variables are multiplied together $(y \cdot y \cdot y \cdot y \cdot y)$. An expression such as $17x$ that has only *one* variable is called *first* degree, and a constant such as 17 which has *no* variable is called *zero* degree.

DEFINITION

DEGREE OF A POLYNOMIAL
The **degree** of a polynomial is the maximum number of variables that appear as factors in any one term.

Various degrees are given special names, as follows:

NAMES

Degree	Name	Example	Memory Aid
0	constant	13	Constants do not vary.
1st	linear	$5x$	A line has *one* dimension.
2nd	quadratic	$7x^2$	A square is a quadrangle.
3rd	cubic	$4x^3$	A cube has *three* dimensions.
4th	quartic	x^4	A *quart* is a *fourth* of a gallon.
5th	quintic	$9x^5$	Quintuplets are *five* children.
6th or more	(no special name)	$3x^{17}$	(Make up your *own* names, Hectic, Septic, etc.)

Notes:

1. Various parts of a monomial such as $3x^2$ have special names.

   3 is the *numerical coefficient*.
   x is the *base*.
   2 is the *exponent*.
   x^2 is a *power* (the second power of x).

2. "Zero" could have *any* degree, because 0 equals $0x^3$, $0x^{15}$, $0x^{1066}$, etc. To avoid this difficulty, 0 is usually called a polynomial with *no* degree.

Multiplying Binomials: Multiplying binomials requires a *double* use of the distributive property.

For example,

$$(x - 3)(2x + 5)$$

can be thought of as

$$\text{number} \times (2x + 5).$$

Distributing the "number," you get

$$\text{number} \times 2x + \text{number} \times 5.$$

Recalling that "number" is actually $(x - 3)$, you get

$$(x - 3) \times 2x + (x - 3) \times 5.$$

Distributing the $2x$ and the 5 gives

$$2x^2 - 6x + 5x - 15,$$

which can be simplified by combining like terms to give

$$2x^2 - x - 15.$$

Once you understand the procedure, you can multiply two binomials *quickly,* in your head. Just multiply each term of one binomial by each term of the other, and write down the answer.

The following exercise is designed to give you practice identifying and naming polynomials, and multiplying binomials.

EXERCISE 1-4

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Evaluate: $23 - 3 \cdot 7$

Q2. Evaluate $|12x - 4|$ if x is -2.

Q3. Distribute the 0.5: $0.5(8x - 9t)$

Q4. Find 70% of 700.

Q5. Subtract and simplify: $\frac{2}{9} - \frac{8}{9}$

Q6. What axiom was used? $(3 + x) + 4 = (x + 3) + 4$

Q7. What axiom tells that 2 times 3 is a unique real number?

Q8. Is $\sqrt{48}$ a rational number?

Q9. Square 7.

Q10. Write an expression representing 17 more than x.

For Problems 1 through 24, tell whether or not the given expression is a polynomial. If it *is,* then name it according to its *degree* and number of *terms* (e.g., "quadratic trinomial"). If it is *not* a polynomial, then tell *why* not.

1. $3x^4 - 2x + 51$

2. $3^2x^3 + 5y^4$

3. $9xy + 2z$
4. $9xyz + 2$
5. $3x^2y - \frac{57}{x}$
6. $3x^2y - \sqrt{57x}$
7. $\frac{x}{2} - 1$
8. $\sqrt{x} \times 5$
9. $\frac{2}{x} - 1$
10. $x + \sqrt{5}$
11. -13
12. 19
13. $8x^3 + 5x^2 - 2x + 11$
14. $3x^2 + 5x - 7$
15. $x^3y^2 - \pi$
16. $xy^5 - \pi^7$
17. $\sqrt{x} + 11$
18. $\frac{x}{4} - 17$
19. $x + \sqrt{11}$
20. $\frac{4}{x} - 17$
21. $5^2x^3y^7 + 8z^9$
22. $6^3x^5y^2 - 3z^6$
23. 0
24. $0x^5$

For Problems 25 through 34, multiply the two binomials.

25. $(x - 3)(x + 7)$
26. $(x - 6)(x + 5)$
27. $(x + 4)(2x - 1)$
28. $(3x + 1)(x - 2)$
29. $(3x - 8)(2x - 7)$
30. $(4x - 3)(7x - 5)$
31. $(2x - 5)(2x - 5)$
32. $(3x - 10)(3x - 10)$
33. $(2x - 5)^2$
34. $(3x - 10)^2$

1-5 EQUATIONS

Evaluating an expression may be thought of as finding out what number the expression equals when you know the value of the variable. You are ready to *reverse* the process, and find out the value of the *variable* when you know what number the *expression* equals. For example, if

$$3x - 5 = 16,$$

then x must equal 7, since $3 \times 7 - 5$ equals 16. The statement "$3x - 5 = 16$" is an equation, and the process of writing down what x