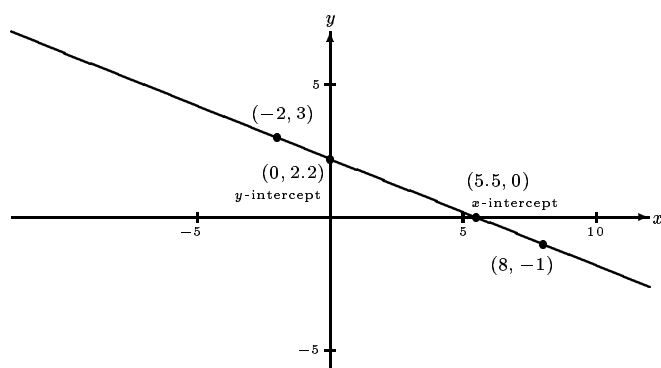# Answers to the Sample of Typical Final Examination Problems

Math 130 Precalculus for the May 20, 2016 Final Exam

1. $y = -\frac{2}{5}x + \frac{11}{5}$ or $y = -0.4x + 2.2$

Also $y - 3 = -\frac{2}{5}(x + 2)$ or $y + 1 = -\frac{2}{5}(x - 8)$

Also $2x + 5y = 11$.

Work: $m = \frac{-1-3}{8-(-2)} = \frac{-4}{10} = -2/5$ and $y - 3 = -\frac{2}{5}(x + 2)$.

2. $\dfrac{f(x+h) - f(x)}{h} = \dfrac{(2x^2 + 4xh + 2h^2 + 3x + 3h - 1) - (2x^2 + 3x - 1)}{h} = \dfrac{4xh + 2h^2 + 3h}{h} = 4x + 2h + 3,\ h \neq 0.$

3. (a) $(f \circ g)(x) = |x^2 - 3|$ (b) $(g \circ f)(x) = x^2 - 6x + 9$

   Domains of f and $g \circ f$: all real numbers x such that $x \geq 0$

   Domains of g and $f \circ g$: all real numbers

   Work for $f \circ g$:

   $x^4 - 6x^2 + 9 = (x^2 - 3)^2$. Then $\sqrt{(x^2 - 3)^2}$ must have a positive answer.(It's the *positive* square root, after all.) That is accomplished by taking the absolute value of $x^2 - 3$. Writing the answer as $x^2 - 3$ is wrong. It is possible for x to be positive and still have $x^2 - 3$ turn out to be negative. For example, consider what happens when $x = 1$: $g(1) = 4$, $f(4) = 2$. But if you had said that the answer was $x^2 - 3$, then $(f \circ g)(1) = 1^2 - 3 = -2$, and that is incorrect.

   The domain of $f \circ g$ is all reals because $x^2 - 6x^2 + 9$, being a perfect square, is always greater than or equal to zero—no matter what the value of x.

4. $f^{-1}(x) = \dfrac{3x-4}{5x}$. It was found by solving $x = \dfrac{4}{-5y+3}$.

As a verbal string it is divide by 4; take reciprocal; subtract 3; change sign; divide by 5. This is the formula
$$-\frac{1}{5}\left(\frac{4}{x}-3\right).$$

The two answers, while different formulas, are algebraically equivalent.

The orginal verbal string was (before inversion) multiply by 5, change sign, add 3, take reciprocal, multiply by 4. All that had to be done was to find the inverse of each operation and apply them in the opposite order as they were in the original function.
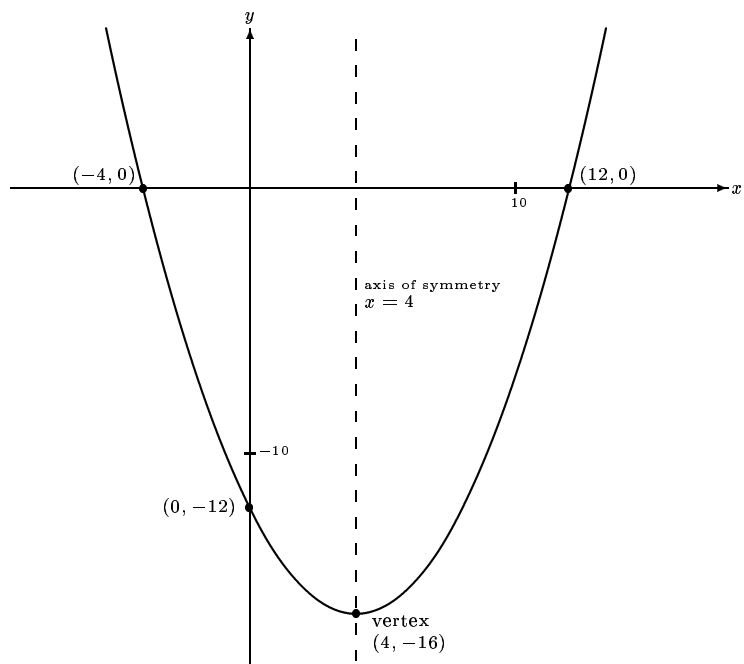
Because this problem is not easily conceptualized, one should apply the two results to a number of his choosing to see if they work.

Let $x = 3$, so $f(3) = \frac{4}{-15+3} = \frac{4}{-12} = -\frac{1}{3}$.

Then $f^{-1}(-1/3) = \frac{3(-1/3)-4}{5(-1/3)} = \frac{-5}{-5/3} = 3$. It checks.

And the verbal: $-1/3$ divided by 4 is $-1/12$, reciprocal is -12, subtract 3 gives -15, change sign gives 15, divide by 5 gives 3, as expected.

5. $\qquad\qquad f(x) = \frac{1}{4}(x-4)^2 - 16.$

6. Let x = the width and let z = the length. The area = xz.
Since $x + z = 26$, we have $z = 26 - x$. Substituting $26 - x$ for z, we find that the area = $xz = x(26 - x) = 26x - x^2 = -x^2 + 26x$. So for the area we have $f(x) = -x^2 + 26x$ with $a = -1 < 0$. That means that the parbola opens down and has a maximum value of $f(x)$.

$$k = c - \frac{b^2}{4a} = 0 - \frac{26^2}{4(-1)} = -\frac{576}{-4} = \frac{-576}{-4} = 169 \text{ square feet}$$

The maximum area is equal to the maximum value of $f(x)$, which equals the maximum value of y, which is called k. Remember that the quantity to be maximized is represented by y.
This problem could also be done by completing the square the long way.
We find that $f(x) = -x^2 + 26x = -(x^2 - 26x) = -(x^2 - 26x + 169 - 169) = -(x - 13)^2 + 169$.
Thus $f(x) = -(x - 13)^2 + 169$ and the vertex is $(13, 169) = (h, k)$.

7. (a) $4w$ (b) $2w + 2$ (c) $2w + 4$ (d) $w/4$ (e) $w^2 + 4w + 4$ (f) $\sqrt{2w}$

8. (a) 2.838   (b) −0.8095   (c) 7.6485   (d) 0.87   (e) 4.7455   (f) −6.392   (g) 4.5495   (h) 0.69933

9. (a) 7/3

    (b) It is six more than twice it.

10. Any solution must be a positive number for which $1 - 3x > 0$ also. That means $x < 1/3$, and we have $0 < x < 1/3$. A valid solution must lie in the interval $(0, 1/3)$.

    By the product rule for logarithms: $\log_2(x - 3x^2) = -4$.

    Rewriting in exponent form: $2^{-4} = x - 3x^2$.

    Then $3x^2 - x + \frac{1}{16} = 0$.

    This could be solved by the quadratic formula, or by multiplying out by 16 then factoring, or by straight factoring using fractions.

    Quadratic formula: $\dfrac{1 \pm \sqrt{1 - 4(3)(1/16)}}{6} = \dfrac{1 \pm \frac{1}{2}}{6}$.

    The answers are $\frac{3/2}{6} = \frac{3}{12} = 1/4$ and $\frac{1/2}{6} = 1/12$.

    Multiply out: $48x^2 - 16x + 1 = 0$, factor as $(12x - 1)(4x - 1) = 0$, so $12x - 1 = 0$ and $x = 1/12$,
    or $4x - 1 = 0$ and $x = 1/4$.

    Simple factoring with fractions: $(3x - 1/4)(x - 1/4) = 0$ will give $3x - 1/4 = 0$, $x = 1/12$,
    and $x - 1/4 = 0$, $x = 1/4$.

    Check when $x = 1/4$:
    $\log_2(1/4) + \log_2(1 - 3(1/4)) = -2 + \log_2(1/4) = -2 + (-2) = -4$; it's OK.

11. Any solution must be a positive number for which $1 - 2x > 0$ also. That means $x < 1/2$, and we have $0 < x < 1/2$. A valid solution must lie in the interval $(0, 1/2)$.

    By the product rule for logarithms: $\log_2(x - 2x^2) = -3$.

    Rewriting in exponent form: $2^{-3} = x - 2x^2$.

    Then $2x^2 - x + \frac{1}{8} = 0$.

    This could be solved by the quadratic formula, or by multiplying out by 8 then factoring, or by straight factoring using fractions.

    Quadratic formula: $\dfrac{1 \pm \sqrt{1 - 4(2)(1/8)}}{4} = \dfrac{1 \pm 0}{4} = 1/4$.

    Multiply out: $16x^2 - 8x + 1 = 0$, perfect square $(4x - 1)^2 = 0$, so $4x - 1 = 0$ and $x = 1/4$.

    Simple factoring with fractions: $(2x - 1/2)(x - 1/4) = 0$ will give $2x - 1/2 = 0$, $x = 1/4$,
    and $x - 1/4 = 0$, $x = 1/4$.

    Check:

    $\log_2(1/4) + \log_2(1 - 2(1/4)) = -2 + \log_2(1/2) = -2 + (-1) = -3$; it's OK.

12. (a) True.

    $\log_a uv = \log_a u + \log_a v$

    This is the first property of logarithms.

    (b) False.

    $(\log_a u) \div (\log_a v) = \log_a(u - v)$

    $2 = \log 100 \div \log 10 \neq \log(100 - 10) = \log 90 \approx 1.95$

13. If $4^{m/n}$ is close to 7, then 4^m will be close to 7^n.

    Powers of 4 are: 4, 16, 64, 256, 1024, 4096, 16384, 65536.

    Powers of 7 are: 7, 49, 343, 2401, 16807.

    It looks like the best match is 16384 to 16807 as the ratio will be somewhat close to 1.

    That means 4^7 is close to 7^5 and the best fraction is for the power of 4 will be $7/5 = 1.4$.

    With $4^{1.4} \approx 7$, the equation can be rewritten as $\log_4 7 \approx 1.4$.

14. (a) $\frac{3}{8}\pi$ or $67.5°$.
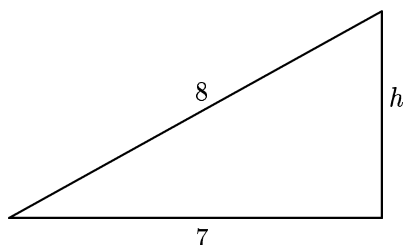
    (b) $157.5°$

    (c) $\frac{11\pi}{15}$.

    (d) 125 feet, because $150° = \frac{5\pi}{6}$ and $\frac{150}{\pi}\left(\frac{5\pi}{6}\right)$ feet $= 125$ feet after cancellation.

15. $\sin\theta = \dfrac{\sqrt{15}}{8}$ $\cos\theta = \dfrac{7}{8}$ $\tan\theta = \dfrac{\sqrt{15}}{7}$ $\csc\theta = \dfrac{8\sqrt{15}}{15}$ $\cot\theta = \dfrac{7\sqrt{15}}{15}$

$\sin 2\theta = \frac{7}{32}\sqrt{15}$ $\cos 2\theta = 17/32$

To find the third side, use $h^2 + 7^2 = 8^2$ so $h = \sqrt{64 - 49} = \sqrt{15}$.

Then $\sec(90° - \theta) = 1/\cos(90° - \theta) = 1/\sin\theta = \csc\theta = \frac{8\sqrt{15}}{15}$.

And $\csc^2\theta - 1 = \cot^2\theta = \left(\frac{7}{\sqrt{15}}\right)^2 = 49/15$.

16. The answers are: $\sin 3\theta = 117/125$ and $\cos 3\theta = 44/125$.

Work: $\sin^2\theta + \cos^2\theta = 1$,   so $\sin^2\theta + 16/25 = 1$, $\sin^2\theta = 9/25$, $\sin\theta = \pm 3/5$.

With the angle (or arc) in the second quadrant, the sine will be positive, making $\sin\theta = 3/5$.

Double angles: $\sin 2\theta = 2\sin\theta\cos\theta = 2(3/5)(-4/5) = -24/25$.

$\cos 2\theta = \cos^2\theta - \sin^2\theta = (-4/5)^2 - (3/5)^2 = 16/25 - 9/25 = 7/25$.

Triple angles: $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta = (-24/25)(-4/5) + (7/25)(3/5) = 96/125 + 21/125 = 117/125$.

$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (7/25)(-4/5) - (-24/25)(3/5) = (-28/125) + (72/125) = 44/125$.

17.   (a) $\cos x(1 + \tan x)(1 - \tan x) = \cos x(1 - \tan^2 x) = \cos x(1 - (\sec^2 x - 1)) = \cos x(2 - \sec^2 x) =$

$2\cos x - \dfrac{1}{\cos x} = \dfrac{2\cos^2 x - 1}{\cos x}$. This answer could also be given as $\dfrac{\cos 2x}{\cos x}$.

(b) $\tan x\cos^2 x = (\sin x/\cos x)\cos^2 x = \sin x\cos x = \frac{1}{2}\sin 2x$.

(c) $\cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = (1)(\cos 2x) = \cos 2x$

(d) $\frac{1 + \cot^2 x}{\sin x} = \frac{\csc^2 x}{\sin x} = \csc^2 x\left(\frac{1}{\sin x}\right) = \csc^2 x(\csc x) = \csc^3 x$.

(e) $\frac{\sec x}{\csc x} = \left(\frac{1}{\cos x}\right)\left(\frac{1}{\csc x}\right) = \left(\frac{1}{\cos x}\right)\sin x = \frac{\sin x}{\cos x} = \tan x$.

(f) $\frac{\sec x}{\sin x} = \sec x\left(\frac{1}{\sin x}\right) = \left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right) = \frac{1}{\sin x\cos x} = \frac{1}{(1/2)\sin 2x} = 2\csc 2x$.

18. An equivalent equation in terms of $\cos x$ is obtained by substituting $2\cos^2 x - 1$ for $\cos 2x$.

    $3(2\cos^2 x - 1) = 2\cos^2 x$, so $4\cos^2 = 3$, $\cos^2 x = 3/4$, and $\cos x = \pm\sqrt{3}/2$.

    For $+\sqrt{3}/2$ the answers are $\pi/6$ and $-\pi/6$, which becomes $11\pi/6$ in the interval specified.

    For $-\pi/6$ the primary solution is $5\pi/6$, the supplement of $\pi/6$. The other solution is $-5\pi/6$, which becomes $7\pi/6$ in the interval specified.

    Answers: $\pi/6$, $5\pi/6$, $7\pi/6$, and $11\pi/6$.

19. The third side is 1 long, the angle opposite the side with length $\sqrt{3}$ is $120°$, and the third angle must be $30°$ because the triangle is isosceles. Note that these 3 angles add to $180°$.

    Work: $c^2 = 1^2 + (\sqrt{3})^2 - 2(1)(\sqrt{3})\cos 30° = 1 + 3 - 2\sqrt{3}(\frac{\sqrt{3}}{2}) = 1 + 3 - 3 = 1$, so $c = 1$.

    Then use the fact that this triangle is isosceles to get the other two angles immediately.

20. The third side is 7 long, the angle opposite the side with length $5\sqrt{3}$ is $141.787°$, and the angle opposite the side with length 2 is $8.213°$. Note that these 3 angles add to $180.000°$.

    The longest side is $5\sqrt{3}$, so the largest angle will be between the sides of lengths 2 and 7.

    The shortest side is 2, so the smallest angle will between the sides of lengths 7 and $5\sqrt{3}$.

    Work: $c^2 = 2^2 + (5\sqrt{3})^2 - 2(2)(5\sqrt{3})\cos 30° = 4 + 75 - 20\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 79 - 30 = 49$, so $c = 7$.

    Largest angle: $\cos^2 = \frac{2^2 + 7^2 - (5\sqrt{3})^2}{2\cdot 2\cdot 7} = \frac{53-75}{28} = \frac{-22}{28} = \frac{-11}{14}$.
    Then the angle is the inverse cosine of $-11/14$, which comes out to about $141.787°$ on a calculator.

    Best practice is to find the third angle by the law of cosines, then check that all three add up to $180°$. This will show if errors were made.

    Smallest angle: $\cos^2 c = \frac{7^2 + (5\sqrt{3})^2 - 2^2}{2(7)(5\sqrt{3})} = \frac{49+75-4}{70\sqrt{3}} = 120/(70\sqrt{3})$.

    Then the angle is the inverse cosine of that expression, which comes out to about $8.213°$ on a calculator.

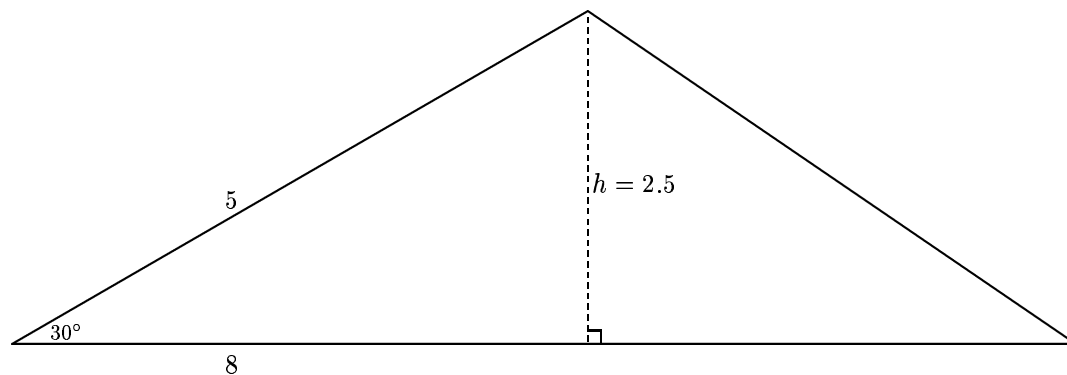   The three angles add up to exactly $180.000°$.
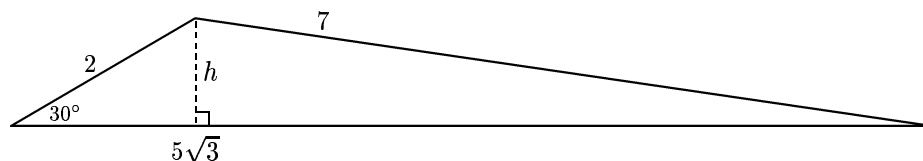
    For the area:
    $\frac{h}{2} = \sin 30° = \frac{1}{2}$.
    So $h = 1$.
    The area $= \frac{1}{2}bh = \frac{1}{2}\cdot 5\sqrt{3}\cdot 1 = 2.5\sqrt{3}$.

21. (a)

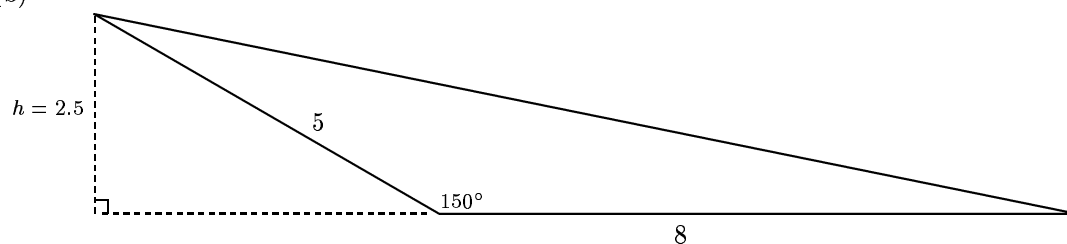The plan: Drop the perpendicular from the uppermost vertex. The trig ratio for the sine of thirty degrees will yield:
$$\sin 30° = \frac{1}{2} = \frac{h}{5}.$$
This may be solved to find that $h = 2.5$.

For the area:
The area of a triangle $= \frac{1}{2}bh = \frac{1}{2} \cdot 8 \cdot 2.5 = 10$.

(b)

The plan: Extend the side of length 8 in the direction of the side of length 5. Then drop the perpendicular from the uppermost vertex. A right triangle is formed with a hypotenuse of length 5. The rightmost angle in the new right triangle is clearly 30 degrees because it and the adjacent angle of 150 degrees add up to 180 degrees. The trig ratio for the sine of thirty degrees will yield:

$$\sin 30° = \frac{1}{2} = \frac{h}{5}.$$
This may be solved to find that $h = 2.5$.

For the area:
The area of a triangle $= \frac{1}{2}bh = \frac{1}{2} \cdot 8 \cdot 2.5 = 10$.
Do not confuse the base which is of length 5 with the extended line.

(c) For the third side c, we have $c^2 = 1^2 + 1^2 - 2(1)(1)(\cos 120°) = 1 + 1 - 2(-\frac{1}{2}) = 2 + 1 = 3$

That means $c^2 = 3$ and $c = \sqrt{3}$.

The other two angles are equal, so each of them measures 30 degrees. That solves the triangle.

An interesting alternative to the Law of Cosines is placing the angle in standard position on the unit circle. The endpoints of the third side are $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(1, 0)$. The distance formula gives $\sqrt{3}$.

(d) The leg opposite the angle of 30 degrees is $1/2$ and the leg opposite the angle of 60 degrees is $\sqrt{3}/2$.

The area equals $\frac{1}{2}bh = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{8}$.

22.

The period is $\dfrac{2\pi}{|a|} = \dfrac{2\pi}{2} = \pi$.

The amplitude is 5.

Factor as $5\sin\left[2\left(x - \frac{\pi}{8}\right)\right]$.

So shift the graph of $y = 5\sin 2x$
$\pi/8$ to the right.