Answers to the Sample of Typical Final Examination Problems

Math 130 Precalculus for the December 18, 2015 Final Exam

1. $(x-11)^2+(y-10.5)^2=7.5$. The center is at $(11,10.5)$ and the radius is 7.5.

   First find the midpoint of the diameter: $(\frac{5+17}{2},\frac{6+15}{2})$. Ths will be the center.

   Then find the distance between the endpoints of the diameter:

   $\sqrt{(17-5)^2+(15-9)^2}=\sqrt{144+81}=\sqrt{225}=15$.

   The radius will be half of that, so 7.5 is the radius.

   The highest point on the circle is at $(11,10.5+7.5)=(11,18)$.

   The point $(5,15)$ is exactly 7.5 from the center by the distance formula:

   $\sqrt{(5-11)^2+(15-10.5)^2}=\sqrt{36+20.25}=\sqrt{56.25}=7.5$. That puts it on the circle.

   The point $(5,17)$ is more than 7.5 from the center by:

   $\sqrt{(5-11)^2+(17-10.5)^2}=\sqrt{36+42.25}=\sqrt{78.25}\approx 8.8459>7.5$. That puts it outside of the circle.

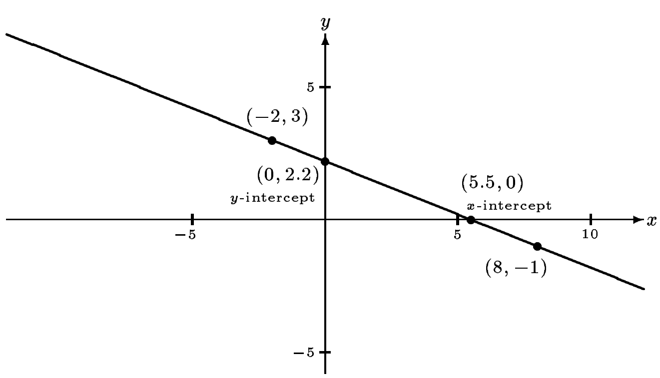
   An alternate purely geometric argument:

   The points $(5,6)$ and $(5,15)$ are both on the circle. Every point on the vertical line segment connecting those two points must be inside the circle; the points themselves are on the circle. That also indicates that evey point on the vertical line $x=5$ below $(5,6)$ or above $(5,15)$ must be outside the circle. The point in question, $(5,17)$, is such a point, so it is situated outside the circle.

2. $y=-\frac{2}{5}x+\frac{22}{5}$ or $y=-0.4x+2.2$

   Also $y-3=-\frac{2}{5}(x+2)$ or $y+1=-\frac{2}{5}(x-8)$

   Also $2x+5y=11$.



   Work: $m=\frac{-1-3}{8-(-2)}=\frac{-4}{10}=-2/5$ and $y-3=-\frac{2}{5}(x+2)$.

3. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{(2x^2+4xh+2h^2+3x+3h-1)-(2x^2+3x-1)}{h}=\dfrac{4xh+2h^2+3h}{h}=4x+2h+3,\ h\neq 0.$

4. (a) $(f \circ g)(x) = |x^2 - 3|$ (b) $(g \circ f)(x) = x^2 - 6x + 9$

Domains of f and $g \circ f$: all real numbers x such that $x \geq 0$

Domains of g and $f \circ g$: all real numbers

Work for $f \circ g$:

$x^4 - 6x^2 + 9 = (x^2 - 3)^2$. Then $\sqrt{(x^2-3)^2}$ must have a positive answer.(It's the *positive* square root, after all.) That is accomplished by taking the absolute value of $x^2 - 3$. Writing the answer as $x^2 - 3$ is wrong. It is possible for x to be positive and still have $x^2 - 3$ turn out to be negative. For example, consider what happens when $x = 1$: $g(1) = 4$, $f(4) = 2$. But if you had said that the answer was $x^2 - 3$, then $(f \circ g)(1) = 1^2 - 3 = -2$, and that is incorrect.

The domain of $f \circ g$ is all reals because $x^2 - 6x^2 + 9$, being a perfect square, is always greater than or equal to zero—no matter what the value of x.

5. $f^{-1}(x) = \dfrac{3x-4}{5x}$. By solving $x = \dfrac{4}{-5y+3}$.

As a verbal string it is divide by 4; take reciprocal; subtract 3; change sign; divide by 5. This is the formula

$$-\frac{1}{5}\left(\frac{4}{x} - 3\right).$$

The two answers, while different formulas, are algebraically equivalent.

The orginal verbal string was (before inversion) multiply by 5, change sign, add 3, take reciprocal, multiply by 4. All that had to be done was to find the inverse of each operation and apply them in the opposite order as they were in the original function.

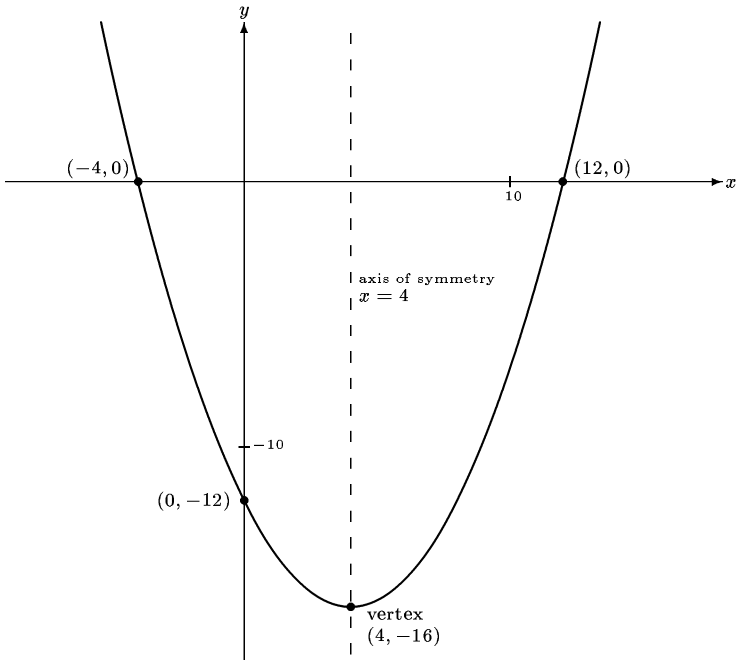
Because this problem is not easily conceptualized, one should apply the two results to a number of his choosing to see if they work.

Let $x = 3$, so $f(3) = \frac{4}{-15+3} = \frac{4}{-12} = -\frac{1}{3}$.

Then $f^{-1}(-1/3) = \frac{3(-1/3)-4}{5(-1/3)} = \frac{-5}{-5/3} = 3$. It checks.

And the verbal: $-1/3$ divided by 4 is $-1/12$, reciprocal is -12, subtract 3 gives -15, change sign gives 15, divide by 5 gives 3, as expected.

6. $f(x) = \frac{1}{4}(x-4)^2 - 16.$



7. Call the two numbers w and x. It is best to avoid using the variable y because of the potential confusion with the actual output.

   The product is $p = wx$. This problem can be reduced to two variables by using the conditions relating w and x linearly.

   $\left(\frac{1}{2}\right) w + x = 5.$

   Solve for w to get $w = 10 - 2x$.

   Substitute in the original equation to get $p = (10-2x)x = 10x - 2x^2$.

   $p = -2x^2 + 10x$ is a quadratic function in x. It opens down and has a maximum.

   The maximum product—the maximum value of this function—is 12.5, obtained from the vertex formula. It occurs when the 2nd number $x = 2.5$, because by the vertex formula $h = -b/2a = -10/(-4) = 2.5$.

   From this result, let us construct the original ordered pair that has the maximum product.

   The product is 12.5, the second number is 2.5, and the first number is $10 - 2(2.5) = 5$, so the ordered pair was $(5, 2.5)$. The product checks as $5 \times 2.5 = 12.5$.

8. (a) $4w$ (b) $2w+2$ (c) $2w+4$ (d) $w/4$ (e) w^2+4w+4 (f) $\sqrt{2w}$

9. (a) 2.838 (b) -0.8095 (c) 7.6485 (d) 0.87 (e) 4.7455 (f) -6.392 (g) 4.5495 (h) 0.69933

10. (a) $7/3$

    (b) It is six more than twice it.

11. (a) $10/3$ (b) $49/16$ or 3.0625

12. (a) $x = 20$ (b) $x = 500/33$ or $x = 15\frac{5}{33}$ (c) $x = 7$ (d) $x = 25$

13. Any solution must be a positive number for which $1 - 2x > 0$ also. That means $x < 1/2$, and we have $0 < x < 1/2$. A valid solution must lie in the interval $(0, 1/2)$.

    By the product rule for logarithms: $\log_2(x - 2x^2) = -3$.
    Rewriting in exponent form: $2^{-3} = x - 2x^2$.
    Then $2x^2 - x + \frac{1}{8} = 0$.
    This could be solved by the quadratic formula, or by mutliplying out by 8 then factoring, or by straight factoring using fractions.

    Quadratic formula: $\dfrac{1 \pm \sqrt{1 - 4(2)(1/8)}}{4} = \dfrac{1 \pm 0}{4} = 1/4$.
    Multiply out: $16x^2 - 8x + 1 = 0$, perfect square $(4x - 1)^2 = 0$, so $4x - 1 = 0$ and $x = 1/4$.
    Simple factoring with fractions: $(2x - 1/2)(x - 1/4) = 0$ will give $2x - 1/2 = 0$, $x = 1/4$, and $x - 1/4 = 0$, $x = 1/4$.

    Check:
    $\log_2(1/4) + \log_2(1 - 2(1/4)) = -2 + \log_2(1/2) = -2 + (-1) = -3$; it's OK.

14. (a) True.
    $\log_a uv = \log_a u + \log_a v$
    This is the first property of logarithms.
    (b) False.
    $(\log_a u) \div (\log_a v) = \log_a(u - v)$
    $2 = \log 100 \div \log 10 \neq \log(100 - 10) = \log 90 \approx 1.95$

15. About 566 bacteria. The one-hour growth factor is $\sqrt[4]{4} = \sqrt{2}$, and $400\sqrt{2} \approx 566$.

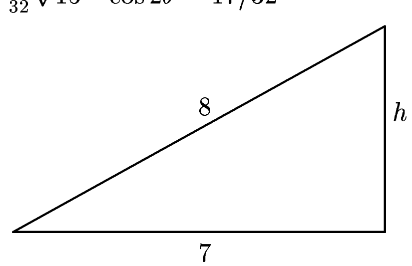
16. About 365 bacteria.
    It takes 6 hours to double, so the base of the exonential function must be $\sqrt[6]{2}$.
    Then $f(4) = 230 \times (\sqrt[6]{2})^4 = 365.10224$.

17. (a) (i) $\dfrac{13\pi}{18}$ (ii) $-\dfrac{\pi}{3}$
    (b) (i) $270°$ (ii) $225°$
    (c) 2.5π meters ≈ 7.85398 meters, because $150° = \frac{5\pi}{6}$ and $3\left(\frac{5\pi}{6}\right)$ meters $= \frac{5\pi}{2}$ meters.

18. (a) 8π rad/min ≈ 25.1327 rad/min (b) 628.31853 ft/min (c) 7.14 miles

19. $\sin\theta = \dfrac{\sqrt{15}}{8}$ $\cos\theta = \dfrac{7}{8}$ $\tan\theta = \dfrac{\sqrt{15}}{7}$ $\csc\theta = \dfrac{8\sqrt{15}}{15}$ $\cot\theta = \dfrac{7\sqrt{15}}{15}$

    $\sin 2\theta = \frac{7}{32}\sqrt{15}$ $\cos 2\theta = 17/32$



To find the third side, use $h^2 + 7^2 = 8^2$ so $h = \sqrt{64 - 49} = \sqrt{15}$.
Then $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\sqrt{15}/8\right)(7/8) = \frac{7}{32}\sqrt{15}$;
$\cos 2\theta = \cos^2\theta - \sin^2\theta = (7/8)^2 - \left(\sqrt{15}/8\right)^2 = 49/64 - 15/64 = 34/64 = 17/32$.
$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (17/32)(7/8) - \left(\frac{7}{32}\sqrt{15}\right)\left(\sqrt{15}/8\right) = 119/256 - 105/256 = 14/256 = 7/128$.

20. The cosine of θ is found from $\cos^2\theta + \sin^2\theta = 1$, $\cos^2\theta + 1/36 = 1$, $\cos^2\theta = 35/36$, $\cos\theta = \sqrt{35}/6$.

    The cosine of θ must be positive as θ is an acute angle in a right triangle and $\cos\theta$ is a trig ratio, not the value of a circular trig function.

    Then $\sin 2\theta = 2\sin\theta\cos\theta = 2(1/6)(\sqrt{35}/6) = \sqrt{35}/18$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta = (\sqrt{35}/6)^2 - (1/6)^2 = 35/36 - 1/36 = 34/36 = 17/18$.

    For the final step, use the formula for the cosine of the double angle with the original angle set as 2θ: $\cos 4\theta = \cos^2 2\theta - \sin^2 2\theta = (17/18)^2 - (\sqrt{35}/18)^2 = 289/324 - 35/324 = 254/324 = 127/162$.

    Check: $\theta = 9.50406822675°$ by using the inverse sine function on the calculator; $4\theta = 38.37627291°$: the cosine of that is about 0.78395. The 5-digit decimal approximation of 127/162 is 0.78395.

21. The answers are: $\sin 3\theta = 117/125$ and $\cos 3\theta = 44/125$.

    Work: $\sin^2\theta + \cos^2\theta = 1$, so $\sin^2\theta + 16/25 = 1$, $\sin^2\theta = 9/25$, $\sin\theta = \pm 3/5$.

    With the angle (or arc) in the second quadrant, the sine will be positive, making $\sin\theta = 3/5$.

    Double angles: $\sin 2\theta = 2\sin\theta\cos\theta = 2(3/5)(-4/5) = -24/25$.

    $\cos 2\theta = \cos^2\theta - \sin^2\theta = (-4/5)^2 - (3/5)^2 = 16/25 - 9/25 = 7/25$.

    Triple angles: $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta = (-25/25)(-4/5) + (7/25)(3/5) = 96/125 + 21/125 = 117/125$.

    $\cos 3\theta = \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (7/25)(-4/5) - (-24/25)(3/5) = (-28/125) + (72/125) = 44/125$.

22. The best idea for this problem is to use a formula for $\cos 2u$ that involves on $\cos u$, no sines. It is $\cos 2u = 2\cos^2 u - 1$.

    First: $\cos 2u = 2\cos^2 u - 1 = 2(3/\sqrt{10})^2 - 1 = 2(9/10) - 1 = 9/5 - 1 = 4/5$.

    Then for $4u$, basing it on the double angle for $2u$: $\cos 4u = 2\cos^2 2u - 1 = 2(4/5)^2 - 1 = 32/25 - 1 = 7/25$.

    Finally, for $8u$, which is double $4u$: $\cos 8u = 2\cos^2 4u - 1 = 2(7/25)^2 - 1 = 98/625 - 1 = 98/625 - 625/625 = -527/625$.

    For the check: $\cos 341.565° \approx 0.948683$ and $3/\sqrt{10} \approx 0.948683$.

    Eight times that angle is 2732.52°, and the cosine will come out to about -0.8432.

    The answer from the double angle formulas was $-527/625 = -0.8432$. They agree.

23. $\sin 3x = \sin(2x + x) = \sin 2x\cos x + \cos 2x\sin x = (2\sin x\cos x)\cos x + (1 - 2\sin^2 x)(\sin x) = 2\sin x(\cos^2 x) + \sin x - 2\sin^3 x = 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x = 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x = -4\sin^3 x + 3\sin x$.

    We needed to substitute $1 - \sin^2 x$ for $\cos^2 x$ to get the answer in terms of the sine only.

    $\cos 3x = \cos(2x + x) = \cos 2x\cos x - \sin 2x\sin x = (2\cos^2 x - 1)(\cos x) - (2\sin x\cos x)\sin x = 2\cos^3 x - \cos x - 2\sin^2 x\cos x = 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x = 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x = 4\cos^3 x - 3\cos x$.

    We needed to substitute $1 - \cos^2 x$ for $\sin^2 x$ to get the answer in terms of the cosine only.

    Choose 20 degrees and set the calculator to degree mode.

    The sine of 20 degrees is 0.342020143324. Take 4 times its cube and subtract it from 3 times the number and you will get 0.866.

    The cosine of 20 degrees is 0.939692620786. Take 4 times its cube and subtract 3 times the number and you will get 0.500.

    That's very good because the sine and cosine of 60 degrees are 0.866 and 0.5.

24. (a) $\cos x(1+\tan x)(1-\tan x) = \cos x(1-\tan^2 x) = \cos x(1-(\sec^2 x - 1)) = \cos x(2-\sec^2 x) =$

$$2\cos x - \frac{1}{\cos x} = \frac{2\cos^2 x - 1}{\cos x}.$$

(b) $\tan x \cos^2 x = (\sin x/\cos x)\cos^2 x = \sin x \cos x = \frac{1}{2}\sin 2x.$

(c) $\cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = (1)(\cos 2x) = \cos 2x$

25. The third side is 1 long, the angle opposite the side with length $\sqrt{3}$ is 120°, and the third angle must be 30° because the triangle is isosceles. Note that these 3 angles add to 180°.

Work: $c^2 = 1^2 + (\sqrt{3})^2 - 2(1)(\sqrt{3})\cos 30° = 1 + 3 - 2\sqrt{3}(\frac{\sqrt{3}}{2}) = 1 + 3 - 3 = 1$, so $c = 1$.

Then use the fact that this triangle is isosceles to get the other two angles immediately.

26. The third side is 7 long, the angle opposite the side with length $5\sqrt{3}$ is 141.787°, and the angle opposite the side with length 2 is 8.213°. Note that these 3 angles add to 180.000°.

The longest side is $5\sqrt{3}$, so the largest angle will be between the sides of lengths 2 and 7.

The shortest side is 2, so the smallest angle will between the sides of lengths 7 and $5\sqrt{3}$.

Work: $c^2 = 2^2 + (5\sqrt{3})^2 - 2(2)(5\sqrt{3})\cos 30° = 4 + 75 - 20\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 79 - 30 = 49$, so $c = 7$.

Largest angle: $\cos^2 = \frac{2^2+7^2-(5\sqrt{3})^2}{2\cdot 2\cdot 7} = \frac{53-75}{28} = \frac{-22}{28} = \frac{-11}{14}$.

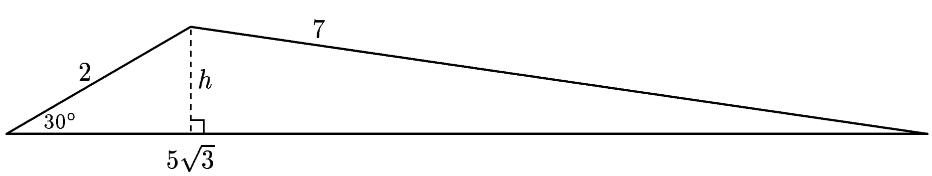
Then the angle is the inverse cosine of $-11/14$, which comes out to about 141.787° on a calculator.

Best practice is to find the third angle by the law of cosines, then check that all three add up to 180°. This will show if errors were made.

Smallest angle: $\cos^2 c = \frac{7^2+(5\sqrt{3})^2-2^2}{2(7)(5\sqrt{3})} = \frac{49+75-4}{70\sqrt{3}} = 120/(70\sqrt{3})$.

Then the angle is the inverse cosine of that expression, which comes out to about 8.213° on a calculator.

The three angles add up to exactly 180.000°.



For the area:

$\frac{h}{2} = \sin 30° = \frac{1}{2}$.

So $h = 1$.

The area $= \frac{1}{2}bh = \frac{1}{2}\cdot 5\sqrt{3}\cdot 1 = 2.5\sqrt{3}$.