From a small number of rather obvious axioms, you will derive all the other properties you will need. In this section you will concentrate on the axioms that apply to the *operations* with numbers such as + and ×. In Section 1-7 you will find the axioms that apply to the *relationships* between numbers, such as = and <.

Objective:

Given the name of an axiom that applies to + or ×, give an example that shows you understand the meaning of the axiom; and vice versa.

There are eleven axioms that apply to adding and multiplying real numbers. These are called the *Field Axioms*, and are listed in the following table. If you already feel familiar with these axioms, you may go right to the problems in Exercise 1-2. If not, then read on!

THE FIELD AXIOMS

CLOSURE

{real numbers} is *closed* under addition and under multiplication. That is, if x and y are real numbers, then

$x + y$ is a ***unique, real*** number,
xy is a ***unique, real*** number.

COMMUTATIVITY

Addition and multiplication of real numbers are *commutative* operations. That is, if x and y are real numbers, then

$x + y$ and $y + x$ are ***equal*** to each other,
xy and yx are equal to each other.

ASSOCIATIVITY

Addition and multiplication of real numbers are *associative* operations. That is, if x, y, and z are real numbers, then

$(x + y) + z$ and $x + (y + z)$ are ***equal*** to each other.
$(xy)z$ and $x(yz)$ are ***equal*** to each other.

DISTRIBUTIVITY

Multiplication *distributes* over addition. That is, if x, y, and z are real numbers, then

$x(y + z)$ and $xy + xz$ are ***equal*** to each other.

IDENTITY ELEMENTS

{real numbers} contains:

A ***unique*** identity element for *addition*, namely 0. (Because $x + 0 = x$ for any real number x.)

A ***unique*** identity element for *multiplication*, namely 1. (Because $x \cdot 1 = x$ for any real number x.)

INVERSES

{real numbers} contains:

A ***unique additive*** inverse for every real number x. (Meaning that every real number x has a real number $-x$ such that $x + (-x) = 0$.)

A ***unique multiplicative*** inverse for every real number x except zero.
(Meaning that every non-zero number x has a real number $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = 1$.)

Notes:

1. Any set that obeys all eleven of these axioms is a *field*.
2. The eleven Field Axioms come in 5 pairs, one of each pair being for addition and the other for multiplication. The Distributive Axiom expresses a relationship between these two operations.
3. The properties $x + 0 = x$ and $x \cdot 1 = x$ are sometimes called the "Addition Property of 0" and the "Multiplication Property of 1," respectively, for obvious reasons.
4. The number $-x$ is called, "the *opposite* of x," "the *additive inverse* of x," or "negative x."
5. The number $\frac{1}{x}$ is called the "multiplicative inverse of x," or the "reciprocal of x."

Closure—By saying that a set is "closed" under an operation, you mean that you cannot get an answer that is *out* of the set by performing that operation on numbers *in* the set. For example, {0, 1} is closed under multiplication because $0 \times 0 = 0$, $0 \times 1 = 0$, $1 \times 0 = 0$, and $1 \times 1 = 1$. All the answers are *unique*, and are *in* the given set. This set is *not* closed under addition because $1 + 1 = 2$, and 2 is *not* in the set. It is not closed under the operation "taking the square root" since there are *two* different square roots of 1: $+1$ and -1.

Commutativity—The word "commute" comes from the Latin word "commutare," which means "to exchange." People who travel back and forth between home and work are called "commuters" because they regularly exchange positions. The fact that addition and multiplication are commutative operations is somewhat unusual. Many operations such as subtraction and exponentiation (raising to powers) are *not* commutative. For example,

$$2 - 5 \text{ does } not \text{ equal } 5 - 2,$$

and

$$2^3 \text{ does } not \text{ equal } 3^2.$$

Indeed, most operations in the real world are not commutative. Putting on your shoes and socks (in that order) produces a far different result from putting on your socks and shoes!

Associativity—You can remember what this axiom states by remembering that to "associate" means to "group." Addition and multiplication are associative, as shown by

$$(2 + 3) + 4 = 9 \quad \text{and} \quad 2 + (3 + 4) = 9.$$

But subtraction is *not* associative. For example,

$$(2 - 3) - 4 = -5 \quad \text{and} \quad 2 - (3 - 4) = 3.$$

Distributivity—Parentheses in an expression such as $2 \times (3 + 4)$ mean, "Do what is inside *first*." But you don't *have* to do $3 + 4$ first. You could "distribute" a 2 to each term inside the parentheses, getting $2 \times 3 + 2 \times 4$. The Distributive Axiom expresses the fact that you get the same answer either way. That is,

$$2 \times (3 + 4) = 14 \quad \text{and} \quad 2 \times 3 + 2 \times 4 = 14.$$

Note that multiplication does *not* distribute over multiplication. For example,

$$2 \times (3 \times 4) \quad \text{does } \textit{not} \text{ equal} \quad 2 \times 3 \times 2 \times 4,$$

as you can easily check by doing the arithmetic.

Identity Elements—The numbers 0 and 1 are called "identity elements" for adding and multiplying, respectively, since a number comes out "identical" if you add 0 or multiply by 1. For example,

$$5 + 0 = 5 \quad \text{and} \quad 5 \times 1 = 5.$$

Inverses—A number is said to be an *inverse* of another number for a certain operation if it "undoes" (or inverts) what the other number did. For example, $\frac{1}{3}$ is the multiplicative inverse of 3. If you start with 5 and multiply by 3 you get

$$5 \times 3 = 15.$$

Multiplying the answer, 15, by $\frac{1}{3}$ gives

$$15 \times \frac{1}{3} = 5,$$

which "undoes" or "inverts" the multiplication by 3. It is easy to tell if two numbers are *multiplicative inverses* of each other because their product is always equal to 1, the multiplicative identity element. For example,

$$3 \times \frac{1}{3} = 1.$$

Similarly, two numbers are *additive inverses* of each other if adding them to each other gives 0, the additive identity element. For example, $\frac{5}{7}$ and $-\frac{5}{7}$ are additive inverses of each other because

$$\tfrac{5}{7} + (-\tfrac{5}{7}) = 0.$$

The following exercise is designed to familiarize you with the names and meanings of the Field Axioms.

EXERCISE 1-2

Do These Quickly

The following problems are intended to refresh your skills. Some problems come from the last section, and others probe your general knowledge of mathematics. You should be able to do all 10 in less than 5 minutes.

Q1. Simplify: $11 - 3 + 5$

Q2. Multiply and simplify: $\left(\frac{2}{3}\right)\left(\frac{6}{7}\right)$

Q3. Add: $3.74 + 5$

Q4. If $x + 7$ is 42, what does x equal?

Q5. Is -13 an integer?

Q6. Multiply: $(9x)(6x)$

Q7. Square 7.

Q8. Is 1.3 a rational number?

Q9. Multiply: $5(3x - 8)$

Q10. Simplify: $(-3)(0.7)(-5)(-1)$

Work the following problems.

1. Tell what is meant by
   a. additive identity element,
   b. multiplicative identity element.
2. What is
   a. the *additive* inverse of $\frac{2}{3}$?
   b. the *multiplicative* inverse of $\frac{2}{3}$?
3. Using variables (x, y, z, etc.) to stand for numbers, write an example of each of the eleven field axioms. Try to do this by writing all eleven

examples first, then checking to be sure you are right. Correct any which you left out or got wrong.

4. Explain why 0 has *no* multiplicative inverse.

5. The Closure Axiom states that you get a *unique* answer when you add two real numbers. What is meant by a "unique" answer?

6. You get the same answer when you add a column of numbers "up" as you do when you add it "down." What axiom(s) show that this is true?

7. Calvin Butterball and Phoebe Small use the distributive property as follows:

   Calvin: $3(x + 4)(x + 7) = (3x + 12)(x + 7)$.
   Phoebe: $3(x + 4)(x + 7) = (3x + 12)(3x + 21)$.

   Who is right? What mistake did the other one make?

8. Write an example which shows that:
   a. Subtraction is *not* a commutative operation.
   b. {negative numbers} is *not* closed under multiplication.
   c. {digits} is *not* closed under addition.
   d. {real numbers} is *not* closed under the $\sqrt{}$ operation (taking the square root).
   e. Exponentiation ("raising to powers") is *not* an associative operation. (Try 4^{2^3}.)

9. For each of the following, tell which of the Field Axioms was used, and whether it was an axiom for *addition* or for *multiplication*. Assume that x, y, and z stand for real numbers.
   a. $x + (y + z) = (x + y) + z$
   b. $x \cdot (y + z)$ is a real number
   c. $x \cdot (y + z) = x \cdot (z + y)$
   d. $x \cdot (y + z) = (y + z) \cdot x$
   e. $x \cdot (y + z) = xy + xz$
   f. $x \cdot (y + z) = x \cdot (y + z) + 0$
   g. $x \cdot (y + z) + (-[x \cdot (y + z)]) = 0$
   h. $x \cdot (y + z) = x \cdot (y + z) \cdot 1$
   i. $x \cdot (y + z) \cdot \dfrac{1}{x \cdot (y + z)} = 1$

10. Tell whether or not the following sets are *fields* under the operations + and ×. If the set is not a field, tell which one(s) of the Field Axioms do not apply.
   a. {rational numbers}
   b. {integers}
   c. {positive numbers}
   d. {non-negative numbers}

Q7. Write the multiplicative identity element.

Q8. If $3x$ equals 42, what does x equal?

Q9. Multiply: (2.3)(4)

Q10. Divide and simplify: $(\frac{2}{3}) \div (\frac{6}{7})$

For Problems 1 through 10, carry out the indicated operations in the agreed-upon order.

1. $5 + 6 \times 7$
2. $3 + 8 \times 7$
3. $9 - 4 + 5$
4. $11 - 6 + 4$
5. $12 \div 3 \times 2$
6. $18 \div 9 \times 2$
7. $7 - 8 \div 2 + 4$
8. $24 - 12 \times 2 + 4$
9. $16 - 4 + 12 \div 6 \times 2$
10. $50 - 30 \times 2 + 8 \div 2$

For Problems 11 through 24, evaluate the given expression
(a) for $x = 2$
(b) for $x = -3$.

11. $4x - 1$
12. $3x - 5$
13. $|3x - 5|$
14. $|4x - 1|$
15. $5 - 7x - 8$
16. $8 - 5x - 2$
17. $|8 - 5x| - 2$
18. $|5 - 7x| - 8$
19. $x^2 - 4x + 6$
20. $x^2 + 6x - 9$
21. $4x^2 - 5x - 11$
22. $5x^2 - 7x + 1$
23. $5 - 2 \cdot x$
24. $3 + 4 \cdot x$

For Problems 25 through 40, simplify the given expression.

25. $6 - [5 - (3 - x)]$
26. $2x - [3x + (x - 2)]$
27. $7(x - 2(3 - x))$
28. $3(6x - 5(x - 1))$
29. $7 - 2[3 - 2(x + 4)]$
30. $8 + 4[5 - 6(x - 2)]$
31. $3x - [2x + (x - 5)]$
32. $4x - [3x - (2x - x)]$
33. $6 - 2[x - 3 - (x + 4) + 3(x - 2)]$
34. $7[2 - 3(x - 4) + 4(x - 6)]$
35. $6[x - \frac{1}{2}(x - 1)]$
36. $8[2x - \frac{1}{4}(6x + 5)]$