1. $\{(x, y): x^2 + y^2 = 25\}$
2. $\{(x, y): x^2 + y^2 + 6x = 16\}$
3. $\{(x, y): x^2 + y^2 - 4y = 21\}$
4. $\{(x, y): x^2 + y^2 + 6x - 4y = 12\}$

9-2 CIRCLES

In Exercise 9-1 you discovered that the graphs of some quadratic relations look like circles. In this section you will learn *why* this is true, and use the results to sketch the graphs *rapidly*.

Objective:

Give the equation or inequality of a circle or circular region, be able to draw the graph *quickly*.

To accomplish this objective, you will start with the geometric definition of a circle, and use the definition to find out what the equation of a circle looks like.

DEFINITION

GEOMETRICAL DEFINITION OF CIRCLE
A **circle** is a set of points in a plane, each of which is equidistant from a *fixed* point called the *center*.

Suppose that a circle of radius r units has its center at the fixed point (h, k) in a Cartesian coordinate system, as shown in Figure 9-2a. If (x, y)

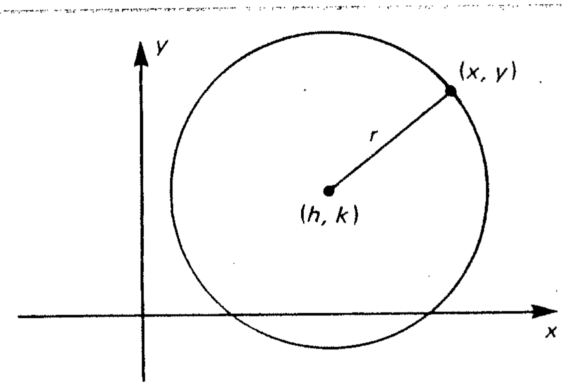


Figure 9-2a

is a point on the circle, then the distance between (x, y) and (h, k) is r units, the radius of the circle. This distance can be expressed in terms of the coordinates of the two points using the Distance Formula.

***Background:** The Distance Formula*

Objective:
Express the distance between two points in a Cartesian coordinate system in terms of the coordinates of the two points.

You recall from your work with slopes of linear functions that the rise and run, Δy and Δx, for the line between the points (x_1, y_1) and (x_2, y_2) are

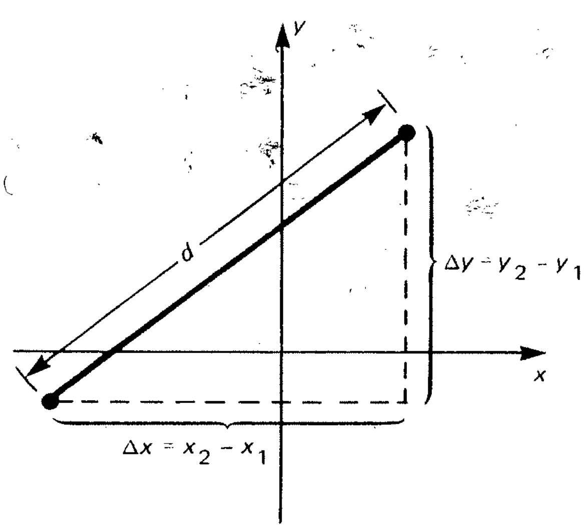
$$\Delta y = y_2 - y_1$$
$$\Delta x = x_2 - x_1.$$

Figure 9-2b shows that Δy, Δx, and the distance d between the two points are the measures of the sides of a right triangle. By Pythagoras, then,

$$d^2 = (\Delta x)^2 + (\Delta y)^2 \quad \text{——} \quad ①.$$

By substitution,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \text{——} \quad ②.$$



The Distance Formula

Figure 9-2b

Taking the positive square root of both members gives

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{——} \quad ③.$$

Equations ①, ②, and ③ are different forms of the *distance formula*. Each form has certain advantages for different applications. The formula is, of course, just a fancy form of the Pythagorean Theorem.

Returning to the problem of finding the equation of a circle, the distance r between the points (x, y) and (h, k) in Figure 9-2a can be expressed using form ② of the distance formula as:

$$\boxed{(x - h)^2 + (y - k)^2 = r^2}$$

where the center is at (h, k), and the radius is r. This is called the *general equation* of a circle.

If the center is at the origin, then $(h, k) = (0, 0)$. In this case the equation reduces to

$$\boxed{x^2 + y^2 = r^2}$$

To show that an equation such as in Problem 4 of Exercise 9-1 is really that of a circle, it is sufficient to transform the equation into the general form, above. This transformation may be accomplished by completing the square.

EXAMPLE

Graph the relation

$$x^2 + y^2 + 6x - 4y - 12 = 0.$$

Solution:
Commuting and associating the terms containing x with each other, and the terms containing y with each other, then adding the constant 12 to both members gives

$$(x^2 + 6x \qquad) + (y^2 - 4y \qquad) = 12.$$

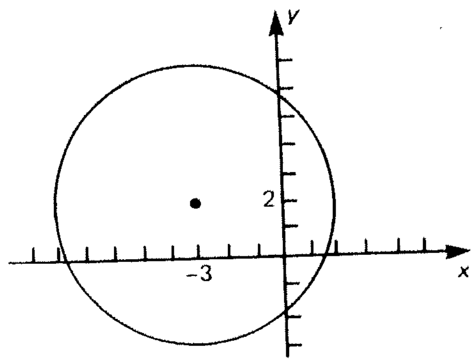
The spaces are left inside the parentheses for completing the square, as you did for equations of parabolas in Chapter 5. Adding 9 and 4 on the left to complete the squares, and on the right to balance the equation, gives

$$(x^2 + 6x + 9) + (y^2 - 4y + 4) = 12 + 9 + 4,$$

from which

$$(x + 3)^2 + (y - 2)^2 = 25.$$

Therefore, the graph will be a *circle* of radius 5 units, centered at $(h, k) = (-3, 2)$, as shown in Figure 9-2c. The graph you drew for Problem 4 of Exercise 9-1 should have looked something like this figure. The



$(x + 3)^2 + (y - 2)^2 = 25$

Figure 9-2c

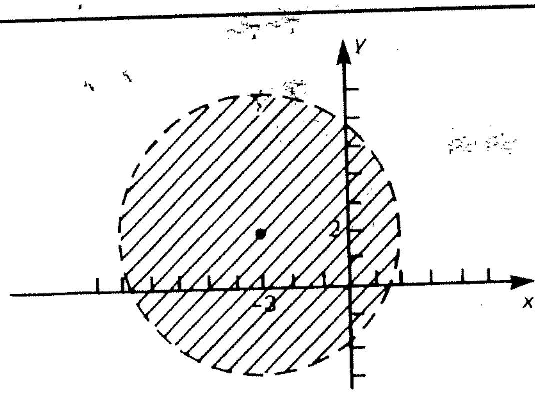
prior knowledge that the graph is a circle of known center and radius, however, will allow you to plot the graph much more quickly, in accordance with your objective for this section. ■

If the open sentence is an *inequality*, then the graph will be the region *inside* the circle if the symbol is "<," as indicated in Figure 9-2d, or *outside* the circle if the symbol is ">."

An objective stated in Section 9-1 is to determine the effects of the constants A, B, C, D, E, and F on the graph of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The above leads to a conclusion about the relative sizes of A and C.



$(x + 3)^2 + (y - 2)^2 < 25$

Figure 9-2d

The graph of a quadratic relation will be a *circle* if the coefficients of the x^2 term and the y^2 term are *equal* (and the xy term is zero).

The following exercise is designed to give you practice drawing graphs of circles and circular regions given the equation or inequality.

EXERCISE 9-2

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Sketch the graph of:

Q1. A linear function.

Q2. A quadratic function.

Q3. A decreasing exponential function.

Q4. An inconsistent linear system.

Q5. A linear inequality.

Q6. A function with a removable discontinuity at $x = 3$.

Q7. A function with a vertical asymptote at $x = 2$.

Write the general equation of:

Q8. A linear function.

Q9. A quadratic function.

Q10. An exponential function.

For Problems 1 through 12, complete the square (if necessary) to find the center and radius. Then draw the graph.

1. $x^2 + y^2 - 10x + 8y + 5 = 0$
2. $x^2 + y^2 + 12x - 2y + 21 = 0$
3. $x^2 + y^2 - 6x - 4y - 12 > 0$
4. $x^2 + y^2 + 16x + 10y - 11 < 0$
5. $x^2 + y^2 - 8x + 6y - 56 \leq 0$
6. $x^2 + y^2 + 4x - 18y + 69 \geq 0$
7. $x^2 + y^2 + 4x - 5 = 0$
8. $x^2 + y^2 - 14y + 48 = 0$

9. $x^2 + y^2 = 49$
10. $x^2 + y^2 = 4$
11. $x^2 + y^2 = 0$ (Watch for a surprise!)
12. $x^2 + y^2 + 16 = 0$ (Watch for a different surprise!)

For Problems 13 through 18, write an equation of the circle described.

13. Center at (7, 5), containing (3, −2)
14. Center at (−4, 6), containing (−2, −3)
15. Center at (−9, −2), containing the origin
16. Center at (5, −4), containing (0, 3)
17. Center at the origin, containing (−6, −8)
18. Center at the origin, containing (−5, 1)
19. Write a general equation for a circle
    a. of radius r, centered at a point on the x-axis,
    b. of radius r, centered at a point on the y-axis,
    c. of radius r, centered at the origin.
20. What do you suppose is meant by a "point circle?" How can you tell from the equation that a circle will be a point circle?
21. Sometimes the graph of a quadratic relation such as you have been plotting turns out to have no points at all! How can you tell from the equation whether or not this will happen?
22. *Introduction to Ellipses* In this section you have found that the graph of a quadratic relation is a circle if x^2 and y^2 have *equal* coefficients. In this problem you will find out what the graph looks like if the coefficients are *not* equal, but have the same sign. Do the following things for the relation whose equation is

$$9x^2 + 25y^2 = 225.$$

    a. Solve the equation for y in terms of x.
    b. Explain why there are *two* values of y for each value of x between −5 and 5.
    c. Explain why there are *no* real values of y for $x > 5$ and for $x < -5$.
    d. Calculate values of y for each integer value of x from −5 to 5, inclusive. Use a calculator or square root tables to approximate any radicals which do not come out integers. Round off to one decimal place.
    e. Plot the graph. If you have been successful, you should have a *closed* figure called an *ellipse*.