Corrected 7/14, 10:47 pm. Property #12 fixed. Thanks, Nour!

We will use these number systems in our work:

Symbol	Name of number system	Definition	Comment	How to write the symbol
N	Natural numbers	(1,2,3,4,5,…}	These are the counting numbers.	$\mathbb{N}$
W	Whole numbers	(0,1,2,3,4,5,…}	The counting numbers, with zero thrown in.	$\mathbb{W}$
Z	Integers	{…, -3,-2,-1,0,1,2, … }	"Z" is the first letter of the German word "Zähl", which means "number". German mathematicians Cantor, Dedekind and others studied the Integers.	$\mathbb{Z}$
Q	Rational Numbers	$\{ p/q \mid p \in Z$, and $q \in Z$, and $q \neq 0\}$. That is, "the set of all fractions with integer numerator, integer denominator, and non-zero denominator.	"Q" stands for "Quotient". A quotient is the answer to a division problem.	$\mathbb{Q}$
R	Real numbers	All numbers on the number line.	Don't confuse "Q" and "R" !!	$\mathbb{R}$
C	Complex numbers	$\{a + b I \mid a \in R$, and $b \in R\}$. Here, the symbol "i" stands for the square root of "-1". All numbers of the form "a + b i" , where "a" and "b" are real numbers, and where "i" represents the square root of "-1".		$\mathbb{C}$

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There are four basic operations on numbers: Addition (+), Multiplication (˙), Subtraction (-), and division (/ or ÷).
These are called the "FIELD PROPERTIES". The Rational numbers, Real numbers, and Complex numbers have these properties. The other number systems have SOME of these properties:

#	Property Name	Definition
1	Closure of addition	If x and y are in the number system, then "x plus y", written "x+y", is also in the number system. x+y is unique.
2	Closure of multiplication	If x and y are in the number system, then "x times y", written "xy". Is also in the number system. xy is unique.
3	Distributivity of multiplication over addition	If x,y, and z are in the number system, then x(y+z) = xy + xz. Note: This is the only property which connects addition and multiplication.
4	Commutativity of addition	If x and y are in the number system, then x+y = y +x .
5	Commutativity of multiplication	If x and y are in the number system, then xy = y x .
6	Associativity of addition	If x,y, and z are in the number system, then (x+y)+z = x + (y + z).
7	Associativity of multiplication	If x,y, and z are in the number system, then (xy)z = x (y z).
8	Additive identity element	There is an element (called "zero" and written "0") such that: If x is in the number system, then x+0 = x, and 0 + x = x.
9	Multiplicative identity element	There is an element (called "one" and written "1") such that: If x is in the number system, then x(1) = x, and 1(x) = x. For clarity: x times 1 equals x, and 1 times x equals x.
10	The number system is non-trivial.	0 ≠ 1. For clarity: Zero is not equal to One. Note: this property seems obvious, but we must have it here.
11	Existence of additive inverses.	If x is in the number system, then there is a number B, in the number system, such that: x + B = B + x = 0. This number B is written "-x". The number (-x) is called **THE ADDITIVE INVERSE of x**. For clarity, x + -x = 0, and -x + x = 0.
12	Existence of multiplicative inverses.	If x is any number **EXCEPT ZERO** in the number system, then there is a number B, in the number system, such that: xB = Bx = 1. This number B is written "1/x" or "x^{-1}". (1/x) is called **THE MULTIPLICATIVE INVERSE of x**. For clarity, x (1/x) = 1, and (1/x) x = 1. ZERO DOES NOT HAVE A MULTIPLICATIVE INVERSE !!!

Definition: A number system that has **all** of these properties is called a **Field**.
Exercise: For the Natural numbers, the Whole numbers, and the Integers, which property or properties above are not true?

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Definition: a number system that has all of these properties except possibly #12, is called a **Commutative Ring with Unity**". An example is the ring of integers, **Z**. Any field is also a commutative rings with unity.

Definition: a **Group** is a number system with only ONE operation, that satisfies properties #2,7,9, and 12 above. The group operation here is multiplication. If also the group also satisfies property #5, then the group is called a **Commutative Group**. A group that is commutative can also be written with the operation "+" rather than "multiplication". In that case it is called an **Additive Group**, and it satisfies properties #1,4,6,8,11.

There are other fields besides **Q**, **R**, and **C**. Here is a simple example:
$\mathbf{Z_2}$ is the field with two elements, named 0 and 1. The addition and multiplication tables for this field are given below:

Addition table for $\mathbf{Z_2}$

+	0	1
0	0	1
1	1	0

This means:
0+0=0; 0+1=1; 1+0=0; 1+1=0.

Multiplication table for $\mathbf{Z_2}$

X	0	1
0	0	0
1	0	1

This means:
0x0=0; 0x1=0; 1x0=0; 1x1=1.

Challenge problem: verify all the field properties for the field Z_2.

Remark: In any field, all the elements of the field, with the single operation of addition, form an Additive Group. Furthermore, in any field, all of the NON-ZERO elements of the field, with the single operation of multiplication, form a commutative group.