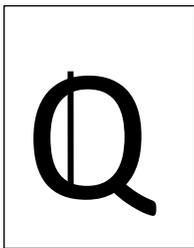
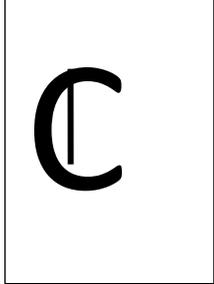


We will use these number systems in our work:

Symbol	Name of number system	Definition	Comment	How to write the symbol
N	Natural numbers	$\{1,2,3,4,5,\dots\}$	These are the counting numbers.	
W	Whole numbers	$\{0,1,2,3,4,5,\dots\}$	The counting numbers, with zero thrown in.	
Z	Integers	$\{\dots, -3,-2,-1,0,1,2, \dots\}$	“Z” is the first letter of the German word “Zähl”, which means “number”. German mathematicians Cantor, Dedekind and others studied the Integers.	
Q	Rational Numbers	$\{p/q \mid p \in \mathbb{Z}, \text{ and } q \in \mathbb{Z}, \text{ and } q \neq 0\}$ . That is, “the set of all fractions with integer numerator, integer denominator, and non-zero denominator.	“Q” stands for “Quotient”. A quotient is the answer to a division problem.	
R	Real numbers	All numbers on the number line.	Don’t confuse “Q” and “R” !!	
C	Complex numbers	$\{a + b i \mid a \in \mathbb{R}, \text{ and } b \in \mathbb{R}\}$ . Here, the symbol “i” stands for the square root of “-1”. All numbers of the form “a + b i”, where “a” and “b” are real numbers, and where “i” represents the square root of “-1”.		

There are four basic operations on numbers: Addition (+), Multiplication ( $\cdot$ ), Subtraction (-), and division ( $/$  or  $\frac{\quad}{\quad}$ ).

These are called the "FIELD PROPERTIES". The Rational numbers, Real numbers, and Complex numbers have these properties. The other number systems have SOME of these properties:

#	Property Name	Definition
1	Closure of addition	If $x$ and $y$ are in the number system, then "x plus y", written " $x+y$ ", is also in the number system. $x+y$ is unique.
2	Closure of multiplication	If $x$ and $y$ are in the number system, then "x times y", written " $xy$ ". Is also in the number system. $xy$ is unique.
3	Distributivity of multiplication over addition	If $x, y,$ and $z$ are in the number system, then $x(y+z) = xy + xz$ . Note: This is the only property which connects addition and multiplication.
4	Commutativity of addition	If $x$ and $y$ are in the number system, then $x+y = y+x$ .
5	Commutativity of multiplication	If $x$ and $y$ are in the number system, then $xy = yx$ .
6	Associativity of addition	If $x, y,$ and $z$ are in the number system, then $(x+y)+z = x + (y+z)$ .
7	Associativity of multiplication	If $x, y,$ and $z$ are in the number system, then $(xy)z = x(yz)$ .
8	Additive identity element	There is an element (called "zero" and written "0") such that: If $x$ is in the number system, then $x+0 = x$ , and $0+x = x$ .
9	Multiplicative identity element	There is an element (called "one" and written "1") such that: If $x$ is in the number system, then $x(1) = x$ , and $1(x) = x$ . For clarity: $x$ times 1 equals $x$ , and 1 times $x$ equals $x$ .
10	The number system is non-trivial.	$0 \neq 1$ . For clarity: Zero is not equal to One. Note: this property seems obvious, but we must have it here.
11	Existence of additive inverses.	If $x$ is in the number system, then there is a number $B$ , in the number system, such that: $x+B = B+x = 0$ . This number $B$ is written " $-x$ ". The number $(-x)$ is called <b>THE ADDITIVE INVERSE of <math>x</math></b> . For clarity, $x + -x = 0$ , and $-x + x = 0$ .
12	Existence of multiplicative inverses.	If $x$ is any number <b>EXCEPT ZERO</b> in the number system, then there is a number $B$ , in the number system, such that: $xB = Bx = 1$ . This number $B$ is written " $1/x$ " or " $x^{-1}$ ". $(1/x)$ is called <b>THE MULTIPLICATIVE INVERSE of <math>x</math></b> . For clarity, $x(1/x) = 1$ , and $(1/x)x = 1$ . <b>ZERO DOES NOT HAVE A MULTIPLICATIVE INVERSE !!!</b>

Definition: A number system that has **all** of these properties is called a **Field**.

Exercise: For the Natural numbers, the Whole numbers, and the Integers, which property or properties above are not true?

Definition: a number system that has all of these properties except possibly #12, is called a **Commutative Ring with Unity**. An example is the ring of integers,  $\mathbf{Z}$ . Any field is also a commutative rings with unity.

Definition: a **Group** is a number system with only ONE operation, that satisfies properties #2,7,9, and 12 above. The group operation here is multiplication. If also the group also satisfies property #5, then the group is called a **Commutative Group**. A group that is commutative can also be written with the operation “+” rather than “multiplication”. In that case it is called an **Additive Group**, and it satisfies properties #1,4,6,8,11.

There are other fields besides  $\mathbf{Q}$ ,  $\mathbf{R}$ , and  $\mathbf{C}$ . Here is a simple example:

$\mathbf{Z}_2$  is the field with two elements, named 0 and 1. The addition and multiplication tables for this field are given below:

+	0	1
0	0	1
1	1	0

This means:  
 $0+0=0$ ;  $0+1=1$ ;  $1+0=0$ ;  $1+1=0$ .

X	0	1
0	0	0
1	0	1

This means:  
 $0x0=0$ ;  $0x1=0$ ;  $1x0=0$ ;  $1x1=1$ .

Challenge problem: verify all the field properties for the field  $\mathbf{Z}_2$ .

Remark: In any field, all the elements of the field, with the single operation of addition, form an Additive Group. Furthermore, in any field, all of the NON-ZERO elements of the field, with the single operation of multiplication, form a commutative group.