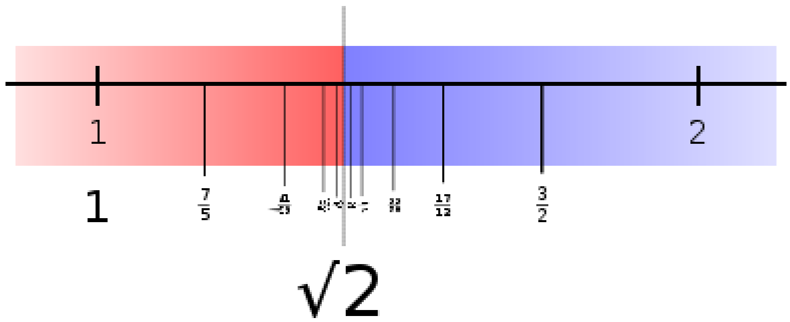
Construction of the Real Number System. Class/Section________Name ______________________
(Corrected 9/4/2014).

1. **Motivation**. The calculus text that we are using has a serious omission: it does not define and construct the Real Numbers **R**. Because of this, a few very important theorems used in Calculus cannot be proved (the book assumes these theorems). One such theorem is the "Intermediate Value Theorem (IVTh)", which states, that if f(x) is a continuous function on the closed interval [a,b], and if f(a) <0 and f(b) >0, then there is some value t $\in$ [a,b], such that f(t) = 0. The IVTh is intuitively obvious, but it can't be proved without a better definition of **R**.
2. **Construction**: We construct **R** as follows:
   Define a ***Dedekind Cut*** as a subset A of the Rational Numbers **Q**, with the following properties:
   a. If $\sim$A is the complement of A in Q, then, for every a $\in$ A, and for every b $\in$ $\sim$A, then a $<$ b.
   b. There is no maximal element of A (that is, there is no w $\in$ A, such that, for any a $\in$ A, a $\leq$ w).
   Then, let **R** = the set of all possible Dedekind Cuts.
3. Picture of √2 as a Dedekind Cut:



4. **Properties of R**:
   If A and B are $\in$ **R**, then we say "A $<$ B" whenever the set A is contained as a subset of B.
   a. Equality: we say A = B as elements of **R**, if A=B as sets.
   b. Addition: we define A+B as the set {x$\in$ **Q** | x = a + b, for some a $\in$ A. and some b $\in$ B}.
   c. Multiplication: define AB as the set {x$\in$ **Q** | x = ab, for some a $\in$ A. and some b $\in$ B}.
   d. Exercise: prove that A+B and AB are Dedekind Cuts.
   e. Embedding of the rational numbers **Q** in **R**: If p is any rational number, then consider the Dedekind Cut P = {x $\in$ **Q** | x $<$ p}.
   f. Exercise: prove that if p and q are rational numbers, then p+q and pq are the appropriate Dedekind Cuts.
   g. Exercise 4f: prove that **R** has all of the properties of a field.
   Example: to show that the associative law of addition holds in **R**, one must prove that if X,Y, and Z are Dedekind Cuts, then (X+Y)+Z = X+(Y+Z). so, first we need to show that both the left and the right-hand side are Dedekind Cuts. (can you do this?). Then, we need to show that the two sides are equal as sets. (can you do that?)
5. **Example: Definition of √2**
   Let S = {s$\in$ **Q** | $s^2 < 2$.}. To prove: $S^2 = 2$.
   Proof: by definition 4c above, S^2 = {x $\in$ **Q** | x=ab, and $a^2<2$, and $b^2<2$}. Since $(ab)^2=a^2b^2<(2)(2)$, then ab<2. So S2 = the Dedekind Cut T ={ x $\in$ **Q** | x<2}, proved.
6. **The field R is a complete ordered field**.
   The set of Dedekind cuts has the least-upper-bound property, i.e., every nonempty subset of it that has any upper bound has a *least* upper bound. This is the **completeness** property. (Proof?) Thus, we have embedded **Q** in a larger field **R**, which is complete.