Proof of the Extreme Value Theorem.

Theorem (Extreme Value Theorem): If f(x) is a function defined and continuous on the closed interval [a,b], then f attains its maximum at some point $c \in [a,b]$.

Outline of proof:

0. Definition. An infinite sequence $< x_n | n \in \mathbb{N} >$ converges to x_0 iff $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, | x_n - x_0 | < \varepsilon$.
1. Bolzano-Weierstrass Theorem. Every bounded infinite sequence has a convergent subsequence.
2. The range of a continuous function f with domain the closed interval [a,b] is bounded. That is, $\exists N \in \mathbb{N}$, such that $\forall x \in [a,b], f(x) < N$.
3. There is a least upper bound for the range of any continuous function defined on a closed interval.
4. The function in #2,3 actually attains its maximum, for some $c \in [a,b]$.

Proof:

Step 1. Bolzano-Weierstrass Theorem. Every bounded infinite sequence has a convergent subsequence.

We may assume that [a,b] = [0,1].
Suppose $<x_n | n \in \mathbb{N}>$ is a sequence of elements on the interval [0,1].
Define a sequence of intervals S_i and points M_i and y_i as follows:
Let $S_1 = [0,1]$. Let M_1 = the midpoint of S_1. $M_1 = ½$. Let $y_1 = x_1$.
Then either $[0, M_1]$ or $[M_1, 1]$ or both, must contain an infinite number of the x_i 's, otherwise the sequence could not be an infinite sequence.
If $[0, M_1]$ contains an infinite number of the x_i's, let $S_2 = [0, M_1]$.
Otherwise, let $S_2 = [M_1,1]$.
Let M_2 = the midpoint of S_2.
Let y_2 = the first x_i beyond x_1 in the sequence $<x_n>$ for which $y_2 \in S_2$.
Continue this process infinitely, choosing if possible the left-hand of the two intervals at each step. Then, $<y_n | n \in \mathbb{N}>$ is a subsequence of $<x_n>$.
From the sequence of left endpoints of S_n, define the number c with a binary expansion
with the n-1th binary place = 0 or 1, according as S_n was the left or right interval chosen.
Claim: the sequence $<y_n | n \in \mathbb{N} >$ converges to the point c.
The claim is true because each interval Sn+1 is contained in the interval Sn, and the width of Sn = $(1/2^n)$. Therefore the sequence $<y_n>$ is a convergent subsequence of $<x_n>$.

Step 2. The range of a continuous function f with domain the closed interval [a,b] is bounded. That is, $\exists N \in \mathbb{N}$, such that $\forall x \in [a,b], f(x) < N$.

Proof: Consider the sequence of sets $T_n = \{x \text{ in } [a,b] \mid f(x) > n\}$, where $n \in \mathbb{N}$.
Each T_n contains T_{n+1}.

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Either all of the T_n are not empty, or there is an N such that $\forall n>N$, T_n is empty.
If all T_n are not empty, then let x_n = the midpoint of T_n.
The sequence $<x_n>$ is an infinite sequence in a bounded closed interval.
By Step (1), the Bolzano-Weierstrass theorem, there exists a convergent subsequence $<y_n>$ of $<x_n>$. Suppose $<y_n>$ converges to c.
Since f is a continuous function, the sequence $< f(y_n) >$ must converge to f(c).
But $f(y_n) > n$. So the limit as n approaches infinity of $f(y_n)$ = infinity. This means f(c) is not
a finite number, which means f is undefined at c, contradicting the assumption that f is defined at every number in [a,b].
Therefore, some T_n is empty. Therefore that number n is an upper bound of the range of f.

Step 3. The range of the function f, in step 2 above, has a least upper bound.
Proof: since the range of f is a subset of the Real Numbers, and it is bounded above, then the range has a least upper bound, by the Completeness property of the Real Numbers.

Step 4. The function in #2,3 actually attains its maximum, for some $c \in [a,b]$.
Proof:
Let L = the upper bound of the range of f. Let $U_n = \{x \in [a,b] \mid f(x) > L - 1/n\}$.
If for some n, U_n were empty, then L could not be the least upper bound of f(x), because $L - 1/n$ would be a smaller upper bound. Therefore, all U_n are non-empty.
For each n, let q_n be an element of U_n . Then, the sequence $<q_n>$ is an infinite sequence, and, by the Bolzano-Weierstrass Theorem, has a convergent subsequence $<r_n>$. For each n, $f(r_n) \geq f(q_n)$ Let c be the limit of the sequence $<r_n>$. Since $f(r_n) \geq f(q_n) > L - 1/n$, the limit as $n \rightarrow$ infinity of $f(q_n)$ must not be less than L. Hence, since f is continuous, f(c)=L, and f attains its maximum at c.

End of proof.