## Review sheet Final Exam Math 140 Calculus I Fall 2015 UMass Boston

The exam is closed textbook. NO CALCULATORS OR ELECTRONIC DEVICES ARE ALLOWED DURING THE EXAM. The final exam will contain problems of types similar to the problem types given here. EACH PROBLEM TYPE given in this review sheet will be REPRESENTED BY AT LEAST ONE EXAM QUESTION. The final will contain 12 or more problems (in turn consisting in a number of sub-problems).

Problem 1. Evaluate the limit if it exists.

 $\lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{x^2 - (x^2 + 2xh + h)^2}{hx^2(x+h)^2}$  $= \lim_{h \to 0} \frac{\cancel{h}(-2x+h)}{\cancel{h}x^2(x+h)^2} = \frac{-2x+0}{x^2(x+0)^2} = -\frac{2}{x^3}.$ 

**Problem 2.** Find the limit or show that it does not exist. If the limit does not exist, indicate whether it is  $\pm \infty$ , or neither. The answer key has not been proofread, use with caution.

Solution. 2.4.

$$\lim_{x \to -\infty} \frac{3x^3 + 2}{2x^3 - 4x + 5} = \lim_{x \to -\infty} \frac{(3x^3 + 2)\frac{1}{x^3}}{(2x^3 - 4x + 5)\frac{1}{x^3}} \qquad \text{Divide top} \\ = \lim_{x \to -\infty} \frac{3 + \frac{2}{x^3}}{2 - \frac{4}{x^2} + \frac{5}{x^3}} \\ = \frac{3 + 0}{2 - 0 + 0} = \frac{3}{2}.$$

Solution. 2.14.

$$\lim_{x \to -\infty} \sqrt{x^2 + x} - \sqrt{x^2 - x} = \lim_{x \to -\infty} \left( \sqrt{x^2 + x} - \sqrt{x^2 - x} \right) \frac{\left(\sqrt{x^2 + x} + \sqrt{x^2 - x}\right)}{\left(\sqrt{x^2 + x} + \sqrt{x^2 - x}\right)}$$
$$= \lim_{x \to -\infty} \frac{x^2 + x - (x^2 - x)}{\sqrt{x^2 + x} + \sqrt{x^2 - x}} = \lim_{x \to -\infty} \frac{2x\frac{1}{x}}{\left(\sqrt{x^2 + x} + \sqrt{x^2 - x}\right)\frac{1}{x}}$$
$$= \lim_{x \to -\infty} \frac{2}{\frac{\sqrt{x^2 + x}}{x} + \frac{\sqrt{x^2 - x}}{x}}} = \lim_{x \to -\infty} \frac{2}{-\sqrt{\frac{x^2 + x}{x^2}} - \sqrt{\frac{x^2 - x}{x^2}}}$$
$$= \lim_{x \to -\infty} \frac{2}{-\sqrt{1 + \frac{1}{x}} - \sqrt{1 - \frac{1}{x}}} = \frac{2}{-\sqrt{1 + 0} - \sqrt{1 - 0}} = -1$$

The sign highlighted in red arises from the fact that, for negative x, we have that  $x = -\sqrt{x^2}$ . **Problem 3.** Evaluate the limit if it exists.

$$1. \lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}.$$

$$2. \lim_{x \to 3} \frac{x^2 - 3x}{x^2 - 2x - 3}.$$

3. 
$$\lim_{x \to 2} \frac{x^2 - 5x - 6}{x - 2}$$
.  
13.  $\lim_{x \to -3} \frac{\frac{1}{3} + 3}{3 + 3}$ 

4. 
$$\lim_{x \to -1} \frac{x^2 - 3x}{x^2 - 2x - 3}$$
.

13. 
$$\lim_{x \to -3} \frac{\frac{1}{3} + \frac{1}{x}}{3+x}$$
.

 $\frac{6}{T}$  = :10MSUD

14. 
$$\lim_{x \to -2} \frac{x^2 + 4x + 4}{x^4 - 16}.$$

15.  $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$ .

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5. 
$$\lim_{x \to -2} \frac{x^2 - 4}{2x^2 + 5x + 2}$$
.

6. 
$$\lim_{x \to -1} \frac{2x^2 + 3x + 1}{3x^2 - 2x - 5}.$$

7. 
$$\lim_{h \to 0} \frac{(-3+h)^2 - 9}{h}$$
.

8. 
$$\lim_{h \to 0} \frac{(-2+h)^3 + 8}{h}$$
.

9. 
$$\lim_{x \to -3} \frac{x+3}{x^3+27}$$
.

10. 
$$\lim_{x \to 1} \frac{x^4 - 1}{x^3 - 1}$$
.

11. 
$$\lim_{h \to 0} \frac{\sqrt{4+h-2}}{h}.$$

12. 
$$\lim_{x \to 3} \frac{\sqrt{5x+1}-4}{x-3}$$
.

4.  $y = \frac{x^2 - 1}{2x^2 - 3x - 2}$ .

17. 
$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{9x - x^2}$$
.

16.  $\lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{x^2 + x} \right).$ 

ANG :JANE

18. 
$$\lim_{h \to 0} \frac{(2+h)^{-1} - 2^{-1}}{h}$$
.

19. 
$$\lim_{x \to 0} \left( \frac{1}{x\sqrt{1+x}} - \frac{1}{x} \right).$$

20. 
$$\lim_{x \to -3} \frac{\sqrt{x^2 + 16} - 5}{x + 3}.$$

21. 
$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$
.

22.  $\lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}.$ 

Z<sup>xE</sup> :Jamsup

 $\frac{G}{E} = :u \partial MSUD$ 

 $\frac{E^x}{7}$  = :10MSUD

Solution. 3.22

$$\lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{hx^2(x+h)^2}$$
$$= \lim_{h \to 0} \frac{\cancel{h}(-2x+h)}{\cancel{h}x^2(x+h)^2} = \frac{-2x+0}{x^2(x+0)^2} = -\frac{2}{x^3}.$$

8 :Jamsuv

Problem 4. Find the horizontal and vertical asymptotes of the graph of the function. Check your work by plotting the function using the Internet.

$$I. \ y = \frac{2x}{\sqrt{x^2 + x + 3} - 3}.$$

$$z - = h \cdot z = h : z = x \cdot z =$$

5. 
$$y = \frac{2x^2 - 2x - 1}{x^2 + x - 2}$$

6.  $y = \frac{1+x^4}{x^2 - x^4}$ .

answer: vertical: x = 1, x = -2, horizontal: y = 2

$$1 - = y$$
 . Norical:  $y = 1$ ,  $x = -1$ ,  $y = 1$ ,  $y = -1$ , horizontal:  $y = -1$ 

answer: vertical: x = 0, no horizontal asymptote

8. 
$$y = \frac{x-9}{\sqrt{4x^2+3x+3}}$$

7.  $y = \frac{x^3 - x}{x^2 - 7x + 6}$ .

answer: no vertical asymptote, horizontal:  $y = \pm \frac{1}{2}$ 

answer: vertical: x = 2,  $x = -\frac{1}{2}$ , horizontal:  $y = \frac{1}{2}$ 

g = y . Nortical: x = 2, horizontal: y = 3

9. 
$$y = \frac{\sqrt{x^2 + 1} - x}{x}$$
.

Solution. 4.1 Vertical asymptotes. A function f(x) has a vertical asymptote at x = a if  $\lim_{x \to \infty} f(x) = \pm \infty$ .

The function is algebraic, and therefore, if it is defined, has a finite limit (i.e., no asymptote). Therefore the function can have vertical asymptotes only for those x for which f(x) is not defined. The function is not defined for  $\sqrt{x^2 + x + 3} - 3 = 0$ , which has two solutions, x = 2 and x = -3. These are precisely the vertical asymptotes: indeed,

$$\lim_{x \to 2^+} \frac{2x}{\sqrt{x^2 + x + 3} - 3} = \infty \qquad \lim_{x \to 2^-} \frac{2x}{\sqrt{x^2 + x + 3} - 3} = -\infty$$
$$\lim_{x \to -3^+} \frac{2x}{\sqrt{x^2 + x + 3} - 3} = \infty \qquad \lim_{x \to -3^-} \frac{2x}{\sqrt{x^2 + x + 3} - 3} = -\infty$$

and

Horizontal asymptotes. A function f(x) has a horizontal asymptote if  $\lim_{x \to \pm \infty} f(x)$  exists. If that limit exists, and is some number, say, N, then y = N is the equation of the corresponding asymptote.

Consider the limit  $x \to -\infty$ . We have that

$$\begin{split} \lim_{x \to -\infty} \frac{2x}{\sqrt{x^2 + 3x + 3} - 3} &= \lim_{x \to -\infty} \frac{2}{\frac{\sqrt{x^2 + x + 3}}{x} - \frac{3}{x}}} \\ &= \lim_{x \to -\infty} \frac{2}{-\sqrt{\frac{x^2 + 3x + 3}{x^2}} - \frac{3}{x}}} \\ &= \lim_{x \to -\infty} \frac{2}{-\sqrt{1 + \frac{3}{x} + \frac{3}{x^2}} - \frac{3}{x}} \\ &= \frac{\lim_{x \to -\infty} 2}{-\sqrt{1 + \frac{3}{x} + \frac{3}{x^2}} - \frac{3}{x}} \\ &= \frac{2}{-\sqrt{1 + \frac{3}{x} + \frac{3}{x^2}} - \frac{3}{x}} \\ &= \frac{2}{-\sqrt{1 + 0 + 0} - 0} \\ &= -2 \end{split}$$

Therefore y = -2 is a horizontal asymptote.

The case  $x \to \infty$ , is handled similarly and yields that y = 2 is a horizontal asymptote. A computer generated graph confirms our computations.



Problem 5. Differentiate.

*1*. tan *x*.

x zoso - :...amsub

4. csc x.

5.  $\sec x \tan x$ .

6.  $\sec x + \tan x$ .

 $\frac{z(\frac{y^2+z^{4c}-1}{2^{4c}-c^{4s-2\psi_{1}}})}{z^{4c}-c^{4s-2\psi_{1}}} = 15. \quad f(x) = \frac{1+x}{1+\frac{2}{2}}.$ 

 $\frac{2}{4} e^{-i\frac{x}{c}} + \frac{9}{2} e^{-i\frac{y}{4}} - 234500}$  16.  $f(x) = \frac{1+x}{1+\frac{3}{x}}$ 8.  $y = ax^2 + bx + c$ . 17.  $f(x) = \frac{x}{x + \frac{c}{2}}$ . 9.  $y = A + \frac{B}{r} + \frac{C}{r^2}$ .

 $\frac{z(z^{x+2})}{zxz}$  :LANSUD  $\frac{e^x}{27-xg}$  :Jamsur

Solution. 6.5 This can be differentiated more effectively using the chain rule, however let us show how the problem can be solved

 $(x \operatorname{sec} x (\operatorname{fgu} x + \operatorname{sec} x))$ 7.  $\sec^2 x$ .  $(x \operatorname{nis} + x \cos x + x \operatorname{nis} x)^x \circ \cdots \circ x + x \operatorname{nis} x)$ answer: 2 tan x sec 2 12.  $\frac{e^x}{\tan x}$ . 8.  $\csc^2 x$ . www.\_\_\_\_\_cof x csc\_\_\_x  $(x_z \cos - x \cos)_x = \cos(x)$ Problem 6. Compute the derivative. 1.  $f(x) = \frac{x-3}{x+3}$ . 10.  $f(t) = \frac{2t}{2 + \sqrt{t}}$ .  $x = (x + \varepsilon)9$  $\frac{2}{2} \left( \frac{\overline{2}}{2} + \overline{2} \right) = \frac{1}{2} \frac{$ 2.  $y = \frac{x^3}{1 - r^2}$ .  $\frac{z(z^{x-1})}{z^{x-z^{x}}} = 11. \quad y = \frac{cx}{1+cx}.$ 3.  $y = \frac{x+1}{x^3 + x - 2}$ .  $z^{-(x_2+1)_2}$  $\frac{z(e^{x+x+z-)}}{e^{xz-z^{x}e-e^{-}}}$  .12.  $y = \sqrt[3]{t}(t^2+t+t^{-1}).$  $\underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}} + \underline{\underline{$  $4. \ y = \frac{x^3 - 2x\sqrt{x}}{r}.$  $\frac{z}{1-x} - xz$  if x - xz if  $y = \frac{u^6 - 2u^3 + 5}{u^2}$ . 5.  $y = \frac{t}{(t-1)^2}$ .  $e^{n_{\overline{v}}} + e^{-n_0t} - z - i u m sup$  $\underbrace{\mathbf{s}^{(1-i)}_{1+i}}_{x+i} \quad 14. \ f(x) = \frac{ax+b}{cx+d}.$ 6.  $y = \frac{t^2 + 2}{t^4 - 3t^2 + 1}$ .  $\frac{z^{(x 2+p)}}{2q-pp}$  : LOMSUD 7.  $g(t) = \frac{t - \sqrt{t}}{t^{\frac{1}{2}}}.$ 

9.  $\frac{\sin x}{x}$ .

10.  $\frac{\sin x}{e^x}$ .

11.  $x \sin x e^x$ .

 $\frac{1}{2} \cos x \cos x \cos x = -\frac{1}{2} \cos x \cos x$ 

answer: sec x tan<sup>2</sup> x + sec<sup>3</sup> x

 $\frac{x^{\partial}}{x \operatorname{uis} - x \operatorname{sod}}$  :...

 $\frac{z^{(x+\varepsilon)}}{z^{+x9+z^x}}$  :.iomsub

directly using the quotient rule.

$$\left(\frac{t}{(t-1)^2}\right)' = \frac{(t)'(t-1)^2 - t\left((t-1)^2\right)'}{(t-1)^4} = \frac{(t-1)^2 - t\left(t^2 - 2t + 1\right)'}{(t-1)^4} = \frac{(t-1)^2 - t\left(2t - 2\right)}{(t-1)^4} = \frac{(t-1)'((t-1) - 2t)}{(t-1)^{\frac{1}{4}} 3} = \frac{-t-1}{(t-1)^3} = -\frac{t+1}{(t-1)^3}$$

**Problem 7.** Compute the derivative using the chain rule.

1.  $f(x) = \sqrt{1+x^2}$ 10.  $y = \cos(4x)$  $\frac{z}{z} - (z + z^{x)x}$  ... (x+) uis + - Jansur 2.  $f(x) = \frac{x}{\sqrt{1 + \frac{2}{x^2}}}$ . 11.  $\sec^2(3x^2)$ .  $\frac{\cos(2x^2)}{\cos(2x^2)^{33}}$  $\frac{z+z^x}{z^{x\mp}}$ 12.  $\csc^2(3x^2)$ . 3.  $y = (\cos x)^{\frac{1}{2}}$  $\frac{\left(\sin\left(2xz\right)\cos\left(x\right)}{\left(\sin\left(2xz\right)\sin\left(x\right)\right)^{3}}$  $\operatorname{uusmol:} \frac{\overline{\mathbf{2}}}{\overline{\mathbf{1}}} x - \frac{\overline{\mathbf{2}}}{\overline{\mathbf{1}}} \cos \left( \underline{\sqrt{x}} \right) = \frac{\overline{\mathbf{2}} \sqrt{\underline{x}}}{\cos \left( \underline{\sqrt{x}} \right)}$ 13.  $2^{3^x}$ . 4.  $f(x) = \sin^3 x$ . (2 ul)(2 ul) x 2 x (1 u 2)(1 u 3) x zuis x soo E :uəmsuv 5.  $y = (1 + \cos x)^2$ . 14.  $3^{2^x}$ .  $x \operatorname{nis} x - 2 \cos x \sin x - 2 \sin x = - \sin(2x) - 2 \sin x$ (E uI)(Z uI) x Z x ZE ::NAWRAND 6.  $f(x) = \frac{1}{\sin^3 x}$ . 15.  $y = \sqrt{\sec(4x)}$  $\frac{x \cos x}{2} = \frac{x \cos x}{2}$ answer:  $2\sqrt{\sec(4x)}$  tan $(4x) = 2(\sec(4x))\frac{3}{2} \sin(4x)$ . 7.  $f(x) = \sqrt[3]{4 + 3\tan x}$ . 16.  $y = x^2 \tan(5x)$ answer:  $(4+3\tan x) - \frac{2}{3} \sec 2 x$ answer:  $2x \tan(5x) - 5x^2 \sec^2(5x)$ 8.  $f(x) = (\cos x + 3\sin x)^4$ . 17.  $y = \frac{1 + \sin(x^2)}{1 + \cos(x^2)}$ .  $(x \operatorname{nis} - x \cos 8)^{\mathbb{E}} (x \operatorname{nis} 8 + x \cos 4) + \cos 6$ 9.  $y = \sin\left(\sqrt{x}\right)$  $\frac{\operatorname{dusmen:}}{\operatorname{dusmen:}} \frac{\operatorname{dusmen:}}{\left(1 + \cos\left(x^2\right)\right)^2} \frac{\left(1 + \cos\left(x^2\right)\right)^2}{\left(1 + \cos\left(x^2\right)\right)^2}$  $\frac{x \wedge z}{\cos x + x}$  : soo Solution. 7.2  $\begin{pmatrix} x \end{pmatrix}'$   $\sqrt{1+\frac{2}{x^2}} - x\left(\sqrt{1+\frac{2}{x^2}}\right)'$   $\sqrt{1+\frac{2}{x^2}} - x\frac{\frac{1}{2}}{\sqrt{1+\frac{2}{x^2}}}\left(\frac{2}{x^2}\right)'$ 

$$\left(\frac{1}{\sqrt{1+\frac{2}{x^2}}}\right) = \frac{\sqrt{1+\frac{2}{x^2}}}{1+\frac{2}{x^2}} = \frac{\sqrt{1+\frac{2}{x^2}}}{1+\frac{2}{x^2}} = \frac{\sqrt{1+\frac{2}{x^2}}}{1+\frac{2}{x^2}} = \frac{x^2\left(1+\frac{2}{x^2}\right)+2}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2+4}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2(1+\frac{2}{x^2})}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2+4}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2(1+\frac{2}{x^2})}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2+4}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2(1+\frac{2}{x^2})}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2+4}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2+4}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2(1+\frac{2}{x^2})}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2(1+\frac{2}{x^2})}{x^2\left(1+\frac{2}{x^2}\right)}$$

Please note that this problem can be solved also by applying the transformation

$$\frac{x}{\sqrt{1+\frac{2}{x^2}}} = \frac{x}{\sqrt{\frac{x^2+2}{x^2}}} = \frac{x}{\frac{1}{\pm x}\sqrt{x^2+2}} = \frac{\pm x^2}{\sqrt{x^2+2}}$$

before differentiating, however one must not forget the  $\pm$  sign arising from  $\sqrt{x^2} = \pm x$ . Our original approach resulted in more algebra, but did not have the disadvantage of dealing with the  $\pm$  sign.

Solution. 7.3

Let 
$$u = \cos x$$
.  
Then  $y = u^{\frac{1}{2}}$ .  
Chain Rule:  $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$   
 $= \left(\frac{1}{2}u^{-\frac{1}{2}}\right)(-\sin x)$   
 $= -\frac{1}{2}\sin x(\cos x)^{-\frac{1}{2}}$ .

Solution. 7.5

Let 
$$u = 1 + \cos x$$
.  
Then  $y = u^2$ .  
Chain Rule:  $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$   
 $= (2u)(-\sin x)$   
 $= -2\sin x(1 + \cos x)$   
 $= -2\sin x - 2\sin x \cos x$   
 $= -2\sin x - \sin(2x)$ . (This last step is optional.)

Solution. 7.9

Let 
$$u = \sqrt{x}$$
.  
Then  $y = \sin u$ .  
Chain Rule:  $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$   
 $= (\cos u)\left(\frac{1}{2}u^{-\frac{1}{2}}\right)$   
 $= \frac{\cos(\sqrt{x})}{2\sqrt{x}}$ .

Solution. 7.15

Chain Rule: 
$$\frac{dy}{dx} = \left(\frac{1}{2}(\sec(4x))^{-\frac{1}{2}}\right) \frac{d}{dx}(\sec(4x))$$
Chain Rule: 
$$\frac{dy}{dx} = \left(\frac{1}{2\sqrt{\sec(4x)}}\right)(\sec(4x)\tan(4x))\frac{d}{dx}(4x)$$

$$= \left(\frac{1}{2\sqrt{\sec(4x)}}\right)(\sec(4x)\tan(4x))(4)$$

$$= \frac{2\sec(4x)\tan(4x)}{\sqrt{\sec(4x)}}$$

There are many ways to simplify this answer, including both of the following.

$$= 2\sqrt{\sec(4x)}\tan(4x).$$
  
= 2(sec(4x))<sup>3/2</sup> sin(4x).

Solution. 7.16

Product Rule: 
$$\frac{dy}{dx} = (x^2)\frac{d}{dx}(\tan(5x)) + (\tan(5x))\frac{d}{dx}(x^2)$$

Use the Chain Rule to differentiate tan(5x) in the first term.

$$\frac{dy}{dx} = (x^2)(-5\sec^2(5x) + (\tan(5x))(2x))$$
$$= 2x\tan(5x) - 5x^2\sec^2(5x).$$

Solution. 7.17

Quotient Rule: 
$$\frac{dy}{dx} = \frac{(1 + \cos(x^2)) \frac{d}{dy} (1 + \sin(x^2)) - (1 + \sin(x^2)) \frac{d}{dx} (1 + \cos(x^2))}{(1 + \cos(x^2))^2}$$

By the Chain Rule,  $\frac{d}{dx}(1 + \cos(x^2)) = -2x\sin(x^2)$  and  $\frac{d}{dx}(1 + \sin(x^2)) = 2x\cos(x^2)$ .

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1+\cos\left(x^2\right))(2x\cos\left(x^2\right)) - (1+\sin\left(x^2\right))(-2x\sin\left(x^2\right))}{(1+\cos\left(x^2\right))^2}$$
$$= \frac{2x\cos\left(x^2\right) + 2x\cos^2\left(x^2\right) + 2x\sin\left(x^2\right) + 2x\sin^2\left(x^2\right)}{(1+\cos\left(x^2\right))^2}$$
$$= \frac{2x(\cos^2\left(x^2\right) + \sin^2\left(x^2\right)) + 2x(\cos\left(x^2\right) + \sin\left(x^2\right))}{(1+\cos\left(x^2\right))^2}$$

By the Pythagorean Identity,  $\cos^2(x^2) + \sin^2(x^2) = 1$ .

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x + 2x(\cos\left(x^2\right) + \sin\left(x^2\right))}{(1 + \cos\left(x^2\right))^2}$$
$$= \frac{2x(1 + \cos\left(x^2\right) + \sin\left(x^2\right))}{(1 + \cos\left(x^2\right))^2}.$$

**Problem 8.** Verify that the coordinates of the given point satisfy the given equation. Use implicit differentiation to find an equation of the tangent line to the curve at the given point. The answer key has not been proofread, use with caution.

 $1. \ y \sin(2x) = x \cos(2y), \left(\frac{\pi}{2}, \frac{\pi}{4}\right).$   $x^{\frac{\pi}{2}} = \hbar \cos x^{\frac{\pi}{2}} = \hbar$ 



First we verify that the point  $(x, y) = \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$  indeed satisfies the given equation:

 $y\sin(2x)_{|x=\frac{\pi}{2},y=\frac{\pi}{4}} = \frac{\pi}{4}\sin\pi = 0 \quad | \text{ left hand side} \\ x\cos(2y)_{|x=\frac{\pi}{2},y=\frac{\pi}{4}} = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 0 \quad | \text{ right hand side}$ 

so the two sides of the equation are equal (both to 0) when  $x = \frac{\pi}{2}$  and  $y = \frac{\pi}{4}$ . Since we are looking an equation of the tangent line, we need to find  $\frac{dy}{dx}|_{x=\frac{\pi}{2},y=\frac{\pi}{4}}$  - that is, the derivative of y at the point  $x = \frac{\pi}{2}$ ,  $y = \frac{\pi}{4}$ . To do so we use implicit differentiation.

$$\begin{array}{rcl} y\sin(2x) &=& x\cos(2y) & | \ \frac{d}{dx} \\ \frac{dy}{dx}\sin(2x) + y\frac{d}{dx}(\sin(2x)) &=& \cos(2y) + x\frac{d}{dx}(\cos(2y)) \\ & \frac{dy}{dx}\sin(2x) + 2y\cos(2x) &=& \cos(2y) - 2x\sin(2y)\frac{dy}{dx} \\ & \frac{dy}{dx}(\sin(2x) + 2x\sin(2y)) &=& \cos(2y) - 2y\cos(2x) & | \ \mathrm{Set} \ x = \frac{\pi}{2}, y = \frac{\pi}{4} \\ & \frac{dy}{dx}_{|x = \frac{\pi}{2}, y = \frac{\pi}{4}} \left(\sin \pi + \pi \sin\left(\frac{\pi}{2}\right)\right) &=& \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\cos \pi \\ & \pi \frac{dy}{dx}_{|x = \frac{\pi}{2}, y = \frac{\pi}{4}} &=& -\frac{\pi}{2}\cos \pi \\ & \frac{dy}{dx}_{|x = \frac{\pi}{2}, y = \frac{\pi}{4}} &=& \frac{1}{2}. \end{array}$$

Therefore the equation of the line through  $x = \frac{\pi}{2}, y = \frac{\pi}{4}$  is

$$y - \frac{\pi}{4} = \frac{1}{2} \left( x - \frac{\pi}{2} \right)$$
$$y = \frac{1}{2} x.$$

1. What is the x-coordinate of the point on the hyperbola  $x^2 - 4y^2 = 16$  that is closest to the point (1,0)? Problem 9.



v = x :....

2. What is the x-coordinate of the point on the ellipse  $x^2 + 4y^2 = 16$  closest to the point (1,0)?

- 3. A rectangular box with a square base is being built out of sheet metal. 2 pieces of sheet will be used for the bottom of the box, and a single piece of sheet metal for the 4 sides and the top of the box. What is the largest possible volume of the resulting box that can be obtained with  $36m^2$  of metal sheet?
- 4. Recall that the volume of a cylinder is computed as the product of the area of its base by its height. Recall also that the surface area of the wall of a cylinder is given by multiplying the perimeter of the base by the height of the cylinder.

A cylindrical container with an open top is being built from metal sheet. The total surface area of metal used must equal  $10m^2$ . Let r denote the radius of the base of the cylinder, and h its height. How should one choose h and r so as to get the maximal possible container volume? What will the resulting container volume be?

## Solution. 9.1

The distance function between an arbitrary point (x, y) and the point (1, 0) is  $d = \sqrt{(x-1)^2 + (y-0)^2}$ . On the other hand, when the point (x, y) lies on the hyperbola we have  $y^2 = \frac{x^2 - 16}{4}$ . In this way, the problem becomes that of minimizing the distance function

$$dist(x) = \sqrt{(x-1)^2 + y^2} = \sqrt{(x-1)^2 + \frac{x^2 - 16}{4}}$$

This is a standard optimization problem: we need to find the critical endpoints, i.e., the points where dist' = 0. As the square root function is an increasing function, the function  $\sqrt{(x-1)^2 + \frac{x^2 - 16}{4}}$  achieves its minimum when the function

$$l = dist^{2} = (x - 1)^{2} + \frac{x^{2} - 16}{4}$$

does. l is a quadratic function of x and we can directly determine its minimimum via elementary methods. Alternatively, we find the critical points of l:

On the other hand,  $x^2 = 16 + 4y^2$  and therefore  $|x| \ge \sqrt{16} = 4$ . Therefore  $x \in (-\infty, -4] \cup [4, \infty)$ . As  $x = \frac{4}{5}$  is outside of the allowed range, it follows that our function either attains its minimum at one of the endpoints  $\pm 4$  or the function has no minimum at all. It is clear however that as x tends to  $\infty$ , so does *dist*. Therefore *dist* attains its minimum for x = 4 or -4 and  $y = \pm \sqrt{(\pm 4)^2 - 16} = 0$ . Direct check shows that  $dist_{|x=4} = \sqrt{(4-1)^2 + \frac{4^2-16}{4}} = 3$  and  $dist_{|x=-4} = \sqrt{(-4-1)^2 + \frac{4^2-16}{4}} = 5$  so our function *dist* has a minimal value of 3 achieved when x = 4, which is our final answer. Notice that this answer can be immediately given without computation by looking at the figure drawn for 9.1. Indeed, it is clear that there are no points from the hyperbola lying inside the dotted circle centered at (1, 0). Therefore the point where this circle touches the hyperbola must have the shortest distance to the center of the circle.

**Solution.** 9.3 Let B denote the area of the base of the box, equal to the area of the top. Let W denote the area of the four walls of the box (the four walls are all equal because the base of the box is a square). Then the surface area S of material used will be

$$S = \underbrace{2B}_{\text{two pieces for the bottom}} + \underbrace{4W}_{4 \text{ walls}} + \underbrace{B}_{\text{the box lid}} = 3B + 4W$$

Let x denote the length of the side of the square base and let y denote the height of the box. Then

$$B = x^2$$

and

$$W = xy$$

As the surface area S is fixed to be 36 square meters, we have that

$$S = 3B + 4W = 36 = 3x^2 + 4xy$$

As y is positive, the above formula shows that  $3x^2 \le 36$  and so  $x \le \sqrt{12}$ . Let us now express y in terms of x:

$$3x^{2} + 4xy = 36$$
  

$$4xy = 36 - 3x^{2}$$
  

$$y = \frac{36 - 3x^{2}}{4x}$$

The problem asks us to maximize the volume V of the box. The volume of the box equals the area of the base times the height of the box:

$$V = B \cdot y = yx^{2} = \frac{(36 - 3x^{2})}{4x}x^{2} = \frac{36x - 3x^{3}}{4}$$

As x is non-negative, it follows that the domain for x is:

$$x \in [0, \sqrt{12}]$$

To maximize the volume we find the critical points, i.e., the values of x for which V' vanishes:

$$0 = V' = \left(\frac{36x - 3x^3}{4}\right)' \\ 0 = \frac{36 - 9x^2}{4} \\ 9x^2 = 36 \\ x^2 = 4 \\ x = \pm 2$$

As x measures length, x = -2 is not possible (outside of the domain for x). Therefore the only critical point is x = 2. Direct check shows that at the endpoints x = 0 and  $x = \sqrt{12}$ , we have that V = 0. Therefore the maximal volume is achieved when x = 2:

$$V_{max} = V_{|x=2} = \frac{36(2) - 3(2)^3}{4} = 12$$
.

Problem 10. Find the

- *the implied domain of f*,
- x and y intercepts of f,
- horizontal and vertical asymptotes,
- intervals of increase and decrease,

Label all relevant points on the graph. Show all of your computations.

- local and global minima, maxima,
- intervals of concavity,
- $f(x) = \frac{x + \frac{1}{2}}{x^2 + x + 1}$   $(p) = \frac{x + \frac{1}{2}}{x^2 + x^2 + \frac{1}{2}}$   $(p) = \frac{x + \frac{1}{2}}{x^2 + x^2 + \frac{1}{2}}$   $(p) = \frac{x + \frac{1}{2}}{x^2 + x^2 + \frac{1}{2}}$   $(p) = \frac{x + \frac{1}{2}}{x^2 + x^2 + \frac{1}{2}}$   $(p) = \frac{x + \frac{1}{2}}{x^2 + \frac{1}{2}}$   $(p) = \frac{x + \frac$

• points of inflection.

3. 
$$f(x) = \frac{2\sqrt{-x^2 + 1} + 1}{\sqrt{-x^2 + 1} + 1}, f(x) = \frac{1}{\sqrt{-x^2 + 1} + 1}$$

The two functions are plotted simultaneously in the x, y-plane. Indicate which part of the graph is the graph of which function.





Solution. 10.2

**Domain.** We have that f is not defined only when we have division by zero, i.e., if  $x^2 - 3x + 3$  equals zero. However, the roots of  $x^2 - 3x + 3$  are not real numbers: they are  $\frac{3\pm\sqrt{3^2-4\cdot 3}}{2} = \frac{3\pm\sqrt{-3}}{2}$ , and therefore  $x^2 - 3x + 3$  can never equal zero. Alternatively, completing the square shows that the denominator is always positive:

$$x^{2} - 3x + 3 = x^{2} - 2 \cdot \frac{3}{2}x + \frac{9}{4} - \frac{9}{4} + 3 = \left(x - \frac{3}{2}\right)^{2} + \frac{3}{4} > 0$$

Therefore the domain of f is all real numbers.

*x*, *y*-intercepts. The *y*-intercept of *f* equals by definition  $f(0) = \frac{2 \cdot 0^2 - 5 \cdot 0 + \frac{9}{2}}{0^2 - 3 \cdot 0 + 3} = \frac{9}{3} = \frac{3}{2}$ . The *x* intercept of *f* is those values of *x* for which f(x) = 0. The graph of *f* shows no such *x*, and that is confirmed by solving the equation f(x) = 0:

$$\begin{aligned} &f(x) &= 0\\ &\frac{2x^2 - 5x + \frac{9}{2}}{x^2 - 3x + 3} &= 0\\ &2x^2 - 5x + \frac{9}{2} &= 0\\ &x_1, x_2 &= \frac{5 \pm \sqrt{25 - 4 \cdot 2 \cdot \frac{9}{2}}}{4} = \frac{5 \pm \sqrt{-9}}{4} \quad , \end{aligned}$$
 Mult. by  $x^2 - 3x + 3$ 

so there are no real solutions (the number  $\sqrt{-9}$  is not real).

Asymptotes. Since f is defined for all real numbers, its graph has no vertical asymptotes. To find the horizontal asymptote(s), we

need to compute the limits  $\lim_{x\to\infty} f(x)$  and  $\lim_{x\to-\infty} f(x)$ . The two limits are equal, as the direct computation below shows:

$$\lim_{x \to \pm \infty} \frac{2x^2 - 5x + \frac{9}{2}}{x^2 - 3x + 3} = \lim_{x \to \pm \infty} \frac{\left(2x^2 - 5x + \frac{9}{2}\right)\frac{1}{x^2}}{\left(x^2 - 3x + 3\right)\frac{1}{x^2}} \\ = \lim_{x \to \pm \infty} \frac{2 - \frac{5}{x} + \frac{9}{2x^2}}{1 - \frac{3}{x} + \frac{3}{x^2}} \\ = \frac{2 - 0 + 0}{1 - 0 + 0} \\ = 2$$

Divide by leading monomial in denominator

Therefore the graph of f(x) has a single horizontal asymptote at y = 2.

Intervals of increase and decrease. The intervals of increase and decrease of f are governed by the sign of f'. We compute:

$$f'(x) = \left(\frac{2x^2 - 5x + \frac{9}{2}}{x^2 - 3x + 3}\right)'$$
  
= 
$$\frac{(2x^2 - 5x + \frac{9}{2})'(x^2 - 3x + 3) - (2x^2 - 5x + \frac{9}{2})(x^2 - 3x + 3)'}{(x^2 - 3x + 3)^2}$$
  
= 
$$\frac{-x^2 + 3x - \frac{3}{2}}{(x^2 - 3x + 3)^2}$$

As the denominator is a square, the sign of f' is governed by the sign of  $-x^2 + 3x - \frac{3}{2}$ . To find where  $-x^2 + 3x - \frac{3}{2}$  changes sign, we compute the zeroes of this expression:

$$\begin{array}{rcl}
-x^2 + 3x - \frac{3}{2} &=& 0 \\
2x^2 - 6x + 3 &=& 0 \\
x_1, x_2 &=& \frac{6 \pm \sqrt{36 - 24}}{4} = \frac{6 \pm \sqrt{12}}{4} \\
x_1, x_2 &=& \frac{3 \pm \sqrt{3}}{2}
\end{array}$$
Mult. by -2

Therefore the quadratic  $-x^2 + 3x - \frac{3}{2}$  factors as

$$-(x - x_1)(x - x_2) = -\left(x - \left(\frac{3 - \sqrt{3}}{2}\right)\right)\left(x - \left(\frac{3 + \sqrt{3}}{2}\right)\right)$$
(1)

The points  $x_1, x_2$  split the real line into three intervals:  $\left(-\infty, \frac{3-\sqrt{3}}{2}\right)$ ,  $\left(\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right)$  and  $\left(\frac{3+\sqrt{3}}{2}, \infty\right)$ , and each of the factors of (1) has constant sign inside each of the intervals. If we choose x to be a very negative number, it follows that  $-(x-x_1)(x-x_2)$  is a negative, and therefore f'(x) is negative for  $x \in (-\infty, \frac{3-\sqrt{3}}{2})$ . For  $x \in (\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2})$ , exactly one factor of f' changes sign and therefore f'(x) is positive in that interval; finally only one factor of f'(x) changes sign in the last interval so f'(x) is negative on  $(\frac{3+\sqrt{3}}{2}, \infty)$ .

Our computations can be summarized in the following table.

Interval	f'(x)	f(x)
$\left(-\infty,\frac{3-\sqrt{3}}{2}\right)$	_	$\searrow$
$\left(\frac{3-\sqrt{3}}{2},\frac{3+\sqrt{3}}{2}\right)$	+	$\nearrow$
$\left(\frac{3+\sqrt{3}}{2},\infty\right)$	_	$\searrow$

Local and global minima and maxima. The table above shows that f(x) changes from decreasing to increasing at  $x = x_1 = \frac{3-\sqrt{3}}{2}$  and therefore f has a local minimum at that point. The table also shows that f(x) changes from increasing to decreasing at  $x = x_2 = \frac{3+\sqrt{3}}{2}$  and therefore f has a local maximum at that point. The so found local maximum and local minimum turn out to be global: there are two things to consider here. First, no other finite point is critical and thus cannot be maximum or minimum - however this leaves out the possibility of a maximum/minimum "at infinity". This possibility can be quickly ruled out by looking at the graph of f. To do so via algebra, compute first  $f(x_1)$  and  $f(x_2)$ :

$$f(x_1) = f\left(\frac{3-\sqrt{3}}{2}\right) = \frac{2\left(\frac{3-\sqrt{3}}{2}\right)^2 - 5\left(\frac{3-\sqrt{3}}{2}\right) + \frac{9}{2}}{\left(\frac{3-\sqrt{3}}{2}\right)^2 - 3\left(\frac{3-\sqrt{3}}{2}\right) + 3} = 2 - \frac{\sqrt{3}}{3}$$
$$f(x_2) = f\left(\frac{3+\sqrt{3}}{2}\right) = \frac{2\left(\frac{3+\sqrt{3}}{2}\right)^2 - 5\left(\frac{3+\sqrt{3}}{2}\right) + \frac{9}{2}}{\left(\frac{3+\sqrt{3}}{2}\right)^2 - 3\left(\frac{3+\sqrt{3}}{2}\right) + 3} = 2 + \frac{\sqrt{3}}{3}$$

On the other hand, while computing the horizontal asymptotes, we established that  $\lim_{x \to \pm \infty} f(x) = 2$ . This implies that all x sufficiently far away from x = 0, we have that f(x) is close to 2. Therefore f(x) is larger than  $f(x_1)$  and smaller than  $f(x_2)$  for all sufficiently far away from x = 0. This rules out the possibility for a maximum or a minimum "at infinity", as claimed above.

**Intervals of concavity.** The intervals of concavity of f are governed by the sign of f''. The second derivative of f is:

$$\begin{aligned} f''(x) &= (f'(x))' = \left(\frac{-x^2 + 3x - \frac{3}{2}}{(x^2 - 3x + 3)^2}\right)' \\ &= \left(-x^2 + 3x - \frac{3}{2}\right)' \left(\frac{1}{(x^2 - 3x + 3)^2}\right) + \left(-x^2 + 3x - \frac{3}{2}\right) \left(\frac{1}{(x^2 - 3x + 3)^2}\right)' \quad \text{second differentiation:} \\ &= (-2x + 3) \left(\frac{1}{(x^2 - 3x + 3)^2}\right) + \left(-x^2 + 3x - \frac{3}{2}\right) (-2) \frac{(x^2 - 3x + 3)'}{(x^2 - 3x + 3)^3} \\ &= (-2x + 3) \left(\frac{1}{(x^2 - 3x + 3)^2}\right) + (2x^2 - 6x + 3) \frac{(2x - 3)}{(x^2 - 3x + 3)^3} \\ &= \frac{(2x - 3)}{(x^2 - 3x + 3)^2} \left(-1 + \frac{(2x^2 - 6x + 3)}{(x^2 - 3x + 3)}\right) \\ &= \frac{(2x - 3)}{(x^2 - 3x + 3)^2} \left(\frac{-(x^2 - 3x + 3) + (2x^2 - 6x + 3)}{(x^2 - 3x + 3)}\right) \\ &= \frac{(2x - 3)(x^2 - 3x)}{(x^2 - 3x + 3)^2} \\ &= \frac{(2x - 3)(x^2 - 3x)}{(x^2 - 3x + 3)^3} \end{aligned}$$

When computing the domain of f, we established that the denominator of the above expression is always positive. Therefore f''(x)changes sign when the terms in the numerator change sign, namely, at x = 0,  $x = \frac{3}{2}$  and x = 3.

Our computations can be summarized in the following table. In the table, we use the  $\cup$  symbol to denote that the function is concave up in the indicated interval, and  $\cap$  to denote that the function is concave down.

Interval	f''(x)	f(x)
$(-\infty,0)$	_	$\cap$
$(0,\frac{3}{2})$	+	U
$(\frac{3}{2},3)$	_	$\cap$
$(3,\infty)$	+	U

**Points of inflection.** The preceding table shows that f''(x) changes sign at  $0, \frac{3}{2}, 3$  and therefore the points of inflection are located at  $x = 0, x = \frac{3}{2}$  and x = 3, i.e., the points of inflection are  $(0, f(0)) = (0, \frac{3}{2}), (\frac{3}{2}, f(\frac{3}{2})) = (\frac{3}{2}, 2), (3, f(3)) = (3, \frac{5}{2}).$ 

We can ask our graphing device to use the so computed information to label the graph of the function. Finally, we can confirm visually that our function does indeed behave in accordance with our computations.



Solution. 10.8

This problem is very similar to Problem 10.2. We recommend to the student to solve the problem first "with closed textbook" and only then to compare with the present solution.

**Domain.** As f is a quotient of two polynomials (rational function), its implied domain is all x except those for which we get division **Domain.** As f is a quotient of two polynomials (rational function), its implied domain is all x except those for which we get division by zero for f. Consequently the domain of f is all x for which  $x^2 + 2x + 4 = 0$ . However, the polynomial  $x^2 + 2x + 4$  has no real roots - its roots are  $\frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{-3}$ , and therefore the domain of f is all real numbers. Alternatively, we can complete the square:  $x^2 + 2x + 4 = (x + 1)^2 + 3$  and so  $x^2 + 2x + 4$  is positive for all values of x. x, y-intercepts. The y-intercept of f equals by definition  $f(0) = \frac{0+1}{0^2 + 2 \cdot 0 + 4} = \frac{1}{4}$ . The x intercept of f is those values of x for

which f(x) = 0. We compute

$$\begin{array}{rcl}
f(x) &=& 0\\ 
\frac{x+1}{x^2+2x+4} &=& 0\\ 
x+1 &=& 0\\ 
x &=& -1
\end{array}$$

and the x-intercept of f is x = -1.

Asymptotes. The line x = a is a vertical asymptote when  $\lim_{x \to \infty} f(x) = \pm \infty$ ; as f is defined for all real numbers, this implies that there are no vertical asymptotes.

The line y = L is a horizontal asymptote if  $\lim_{x \to \pm \infty} f(x)$  exists and equals L. We compute:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{(x+1)\frac{1}{x^2}}{(x^2+2x+4)\frac{1}{x^2}} = \lim_{x \to \infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{4}{x^2}} = \frac{0+0}{1+0+0} = 0$$

Therefore y = 0 is a horizontal asymptote for f. An analogous computation shows that  $\lim_{x \to \pm \infty} f(x) = 0$  and so y = 0 is the only horizontal asymptote of f.

Intervals of increase and decrease. The intervals of increase and decrease of f are governed by the sign of f'. We compute:

$$f'(x) = \left(\frac{x+1}{x^2+2x+4}\right)' \qquad | \text{ qutotient rule} \\ = \frac{(x+1)' (x^2+2x+4) - (x+1) (x^2+2x+4)'}{(x^2+2x+4)^2} \\ = \frac{x^2+2x+4 - (x+1)(2x+2)}{(x^2+2x+4)^2} \\ = \frac{x^2+2x+4 - (2x^2+4x+2)}{(x^2+2x+4)^2} \\ = \frac{-x^2-2x+2}{(x^2+2x+4)^2} \\ = \frac{-x^2-2x+2}{(x^2+2x+4)^2}$$

As  $x^2 + 2x + 4$  is positive, the sign of f' is governed by the sign of  $-x^2 + 2x + 2$ . To find out where  $-x^2 + 2x + 2$  changes sign, we compute the zeroes of this expression:

$$\begin{array}{rcrcrcr} -x^2 - 2x + 2 &= & 0 \\ x^2 + 2x - 2 &= & 0 \\ x_1, x_2 &= & -1 \pm \sqrt{3} \end{array}$$
 | use the quadratic formula

Therefore the quadratic  $-x^2 + 2x + 2$  factors as

$$-(x-x_1)(x-x_2) = -\left(x - \left(-1 - \sqrt{3}\right)\right)\left(x - \left(-1 + \sqrt{3}\right)\right)$$
(2)

The points  $x_1, x_2$  split the real line into three intervals:  $(-\infty, -1-\sqrt{3}), (-1-\sqrt{3}, -1+\sqrt{3})$  and  $(-1+\sqrt{3}, \infty)$ , and each of the factors of (2) has constant sign inside each of the intervals. If we choose x to be a very negative number, it follows that  $-(x - x_1)(x - x_2)$  is a negative, and therefore f'(x) is negative for  $x \in (-\infty, -1 - \sqrt{3})$ . For  $x \in (-1 - \sqrt{3}, -1 + \sqrt{3})$ , exactly one factor of f' changes sign and therefore f'(x) is positive in that interval; finally only one factor of f'(x) changes sign in the last interval so f'(x) is negative on  $(-1 + \sqrt{3}, \infty)$ .

Our computations can be summarized in the following table.

Interval	f'(x)	f(x)
$(-\infty, -1 - \sqrt{3})$	_	$\searrow$
$(-1-\sqrt{3},-1+\sqrt{3})$	+	$\nearrow$
$(-1+\sqrt{3},\infty)$	_	$\searrow$

**Local and global minima and maxima.** The table above shows that f(x) changes from decreasing to increasing at  $x = x_1 =$  $-1 - \sqrt{3}$  and therefore f has a local minimum at that point. The table also shows that f(x) changes from increasing to decreasing at  $x = x_2 = -1 + \sqrt{3}$  and therefore f has a local maximum at that point. The so found local maximum and local minimum turn out to be global: indeed, no other finite point is critical and thus cannot be maximum or minimum; on the other hand  $\lim_{x \to a} f(x) = 1$  and this implies that all x sufficiently far away from x = 0 have that f(x) is close to 0, and therefore f(x) is larger than  $f(x_1)$  and smaller than  $f(x_2)$  for all x.

**Intervals of concavity.** The intervals of concavity of f are governed by the sign of f''. The second derivative of f is:

$$\begin{split} f''(x) &= (f'(x))' = \left(\frac{-x^2 - 2x + 2}{(x^2 + 2x + 4)^2}\right)' \\ &= (-x^2 - 2x + 2)' \left(\frac{1}{(x^2 + 2x + 4)^2}\right) + (-x^2 - 2x + 2) \left(\frac{1}{(x^2 + 2x + 4)^2}\right)' \\ &= (-2x - 2) \left(\frac{1}{(x^2 + 2x + 4)^2}\right) + (-x^2 - 2x + 2)(-2) \frac{(x^2 + 2x + 4)'}{(x^2 + 2x + 4)^3} \\ &= -(2x + 2) \left(\frac{1}{(x^2 + 2x + 4)^2}\right) + (2x^2 + 4x - 4) \frac{(2x + 2)}{(x^2 + 2x + 4)^3} \\ &= \frac{(2x + 2)}{(x^2 + 2x + 4)^2} \left(-1 + \frac{(2x^2 + 4x - 4)}{(x^2 + 2x + 4)}\right) \\ &= \frac{(2x + 2)}{(x^2 + 2x + 4)^2} \left(\frac{-(x^2 + 2x + 4) + (2x^2 + 4x - 4)}{(x^2 + 2x + 4)}\right) \\ &= \frac{(2x + 2)(x^2 + 2x - 8)}{(x^2 + 2x + 4)^3} \\ &= \frac{(2x + 2)(x^2 + 4x - 8)}{(x^2 + 2x + 4)^3} \\ &= \frac{(2x + 2)(x + 4)(x - 2)}{(x^2 + 2x + 4)^3} \end{split}$$

As we previously established, the denominator of the above expression is always positive. Therefore the expression above changes sign when the terms in the numerator change sign, namely, at x = -1, x = -4 and x = 2. Our computations can be summarized in the following table.

Our computat	ions can	De sum
Interval	f''(x)	f(x)
$(-\infty, -4)$	_	$\cap$
(-4, -1)	+	U
(-1,2)	_	$\cap$
$(2,\infty)$	+	U

**Points of inflection.** The preceding table shows that f''(x) changes sign at -4, -1, 2 and therefore the points of inflection are located at x = -4, x = -1 and x = 2, i.e., the points of inflection are  $\left(-4, -\frac{1}{4}\right), \left(-1, 0\right), \left(2, \frac{1}{4}\right)$ .

**Problem 11.** Use the Intermediate Value theorem and the Mean Value Theorem/Rolle's Theorem to prove that the function has **exactly** one real root.

- 1.  $f(x) = x^3 + 4x + 7$ .
- 2.  $f(x) = x^3 + x^2 + x + 1$ .
- 3.  $f(x) = \cos^3\left(\frac{x}{3}\right) + \sin x 3x$ .

**Solution.** 11.1. f(-2) = -9 and f(1) = 12. Since f(x) is continuous and has both negative and positive outputs, it must have a zero. In other words, for some c between -2 and 1, f(c) = 0. If there were solutions x = a and x = b, then we would have f(a) = f(b), and Rolle's Theorem would guarantee that for some x-value, f'(x) = 0. However,  $f'(x) = 3x^2 + 4$ , which always positive and therefore is never 0. Therefore there cannot be 2 or more solutions.

The above can be stated informally as follows. Note that  $f'(x) = 3x^2 + 4$ , which is always positive. Therefore, the graph of f is increasing from left to right. So once the graph crosses the x-axis, it can never turn around and cross again, so there can only be a single zero (that is, a single solution to f(x) = 0).

**Solution.** 11.3.  $f(5) = \cos^3\left(\frac{5}{3}\right) + \sin 5 - 15 \le 2 - 15 = -13 < 0$  (because  $\cos a, \sin b \in [-1, 1]$  for arbitrary a, b). Similarly  $f(-5) = \cos^3\left(-\frac{5}{3}\right) + \sin(-5) + 15 \ge 15 - 2 > 0$ . Therefore by the Intermediate Value Theorem f(x) = 0 has at least one solution in the interval [-5, 5].

Suppose on the contrary to what we are trying to prove, f(x) = 0 has two or more solutions; call the first 2 solutions a, b. That means that f(a) = f(b) = 0, so by the Mean value theorem, there exists a  $c \in (a, b)$  such that f'(c) = (f(a) - f(b))/(a - b) = (0 - 0)/(a - b) = 0. On the other hand we may compute:

$$f'(x) = -3 + \cos x - \cos^2\left(\frac{x}{3}\right)\sin\left(\frac{x}{3}\right) \le -1 < 0,$$

where the first inequality follows from the fact that  $\sin x, \cos x \in [-1, 1]$ . So we got that f'(c) = 0 for some c but at the same time f'(x) < 0 for all x, which is a contradiction. Therefore f(x) = 0 has exactly one solution.

**Problem 12.** Estimate the integral using a Riemann sum using the indicated sample points and interval length.

 $I. \int_{0}^{4} \left(\sqrt{8x+1}\right) dx. \text{ Use four intervals of equal width, choose the sample point to be the left endpoint of each interval.}$   $\underbrace{: \underline{u} + 6 \approx xp(x) \int_{p}^{0} \int_{p}^{smult} \underbrace{: \underline{u} + xg}_{p} = (x) \int_{p}^{1} purt \underline{u} = x\nabla : I \neq N = 0$ 

$$\int_{0}^{6} \frac{1}{x^{2}+1} dx. \text{ Use three intervals of equal width, choose the sample point to be the left endpoint.}$$

$$\frac{\cdot \frac{98}{x_{17}}}{\sqrt{2}} \approx x_{\text{P}}(x) \int_{0}^{0} \sup_{x} \frac{\cdot \frac{1+z^{x}}{1}}{1} = (x) \int_{0}^{y} \sup_{x} z = x_{\nabla} \frac{z_{\text{PMSUP}}}{1}$$

3. 
$$\int_{-3.5}^{-0.5} \frac{dx}{x^2 + 1}$$
. Use three intervals of equal width, choose the sample point to be the midpoint of each interval.

$$S.0 = \frac{4}{5} = ((1-) t + (2-) t + ((1-) t) x \Delta \approx xb(x) t \int_{3.5}^{5.0-} 2u T \cdot \frac{1}{1+2x} = (x) t \text{ bno } 1 = x\Delta$$
 (39) where  $x = x\Delta$  (39) and  $x = x\Delta$  (30) and  $x = x\Delta$ 

4. 
$$\int_0^2 \frac{\mathrm{d}x}{1+x+x^3}$$
. Use  $\Delta x = \frac{1}{2}$  and right endpoint sampling points.

2.

 $\text{answer: } \frac{1}{2} \left( \frac{8}{12} + \frac{1}{2} + \frac{8}{2} + \frac{1}{2} + \frac{8}{12} + \frac{1}{2197} \right) \approx 0.604920$ 

5. 
$$\int_{-2}^{0} \frac{\mathrm{d}x}{1+x+x^2}$$
. Use  $\Delta x = \frac{2}{3}$  and left endpoint sampling points

 $\text{answer: } \frac{2}{3} \left( \frac{1}{3} + \frac{1}{9} + \frac{7}{9} \right) = \frac{1262}{1262} \approx 1.540904$ 

6.  $\int_{0}^{z} \frac{dx}{1+x^3}$ . Use four intervals of equal width, choose the sample point to be the left endpoint of each interval.

$$\text{E7808:} \ \ \text{2} = 0.5 \text{ and } t = 0.5 \text{ and } t = 0 \text{ and } t = 0 \text{ bold} = 0 \text{ bold}$$

7. 
$$\int_{-2}^{0} \frac{\mathrm{d}x}{x^4 + 1}$$
. Use four intervals of equal width, choose the sample point to be the right endpoint.  
$$z_{90000^\circ,\mathrm{T}} \approx \frac{9629}{2628} = \left( (0)f + \left(\frac{z}{\tau}\right) f + (\tau-)f + \left(\frac{z}{\varepsilon}\right) f \right) x_{\nabla} \approx x_{\mathrm{P}}(x) f \int_{\tau}^{0} s_{\mathrm{PL}} \cdot \frac{\varepsilon^{x+1}}{\tau} = (x)f p_{\mathrm{PL}} \cdot \varepsilon \cdot \varepsilon = x_{\nabla} \cdot \frac{1}{2} s_{\mathrm{PL}} \cdot \frac{\varepsilon^{x+1}}{\tau} = (x)f p_{\mathrm{PL}} \cdot \varepsilon = x_{\nabla} \cdot \frac{1}{2} s_{\mathrm{PL}} \cdot \frac{\varepsilon^{x+1}}{\tau} = (x)f p_{\mathrm{PL}} \cdot \frac{\varepsilon^{x+1}}$$

Solution. 12.1. The interval [0, 4] is subdivided into n = 4 intervals, therefore the length of each is  $\Delta x = 1$ . The intervals are therefore

[0,1], [1,2], [2,3], [3,4].

The problem asks us to use the left endpoints of each interval as sampling points. Therefore our sampling points are 0, 1, 2, 3. Therefore the Riemann sum we are looking for is

$$\Delta x \left( f(0) + f(1) + f(2) + f(3) \right) = 1 \cdot \left( \sqrt{8} \cdot 0 + 1 + \sqrt{8} \cdot 1 + 1 + \sqrt{8} \cdot 2 + 1 + \sqrt{8} \cdot 3 + 1 \right) = 9 + \sqrt{17} \approx 13.1231$$

**Solution.** 12.3. The interval [-3.5, -0.5] is subdivided into n = 3 intervals, therefore the length of each is  $\Delta x = 1$ . The intervals are therefore

$$[-3.5, -2.5], [-2.5, -1.5], [-1.5, -0.5]$$
.

The problem asks us to use the midpoint of each interval as a sampling point. Therefore our sampling points are -3, -2, -1. Therefore the Riemann sum we are looking for is

$$\Delta x \left( f(-3) + f(-2) + f(-1) \right) = 1 \cdot \left( \frac{1}{10} + \frac{1}{5} + \frac{1}{2} \right) = 0.8 \quad .$$

 $-\frac{7}{2}$   $-\frac{5}{2}$   $-\frac{3}{2}$   $-\frac{1}{2}$  x

Problem 13. Evaluate the definite integral.

$$\begin{split} I. & \int_{-2}^{3} (x^{2} - 1) \, dx. & 9. \int_{1}^{4} \frac{\sqrt{x}}{\sqrt{x}} + 1 + x}{\sqrt{x}} dx. & 18. \int_{0}^{1} \left| x - \frac{1}{2} \right| dx. \\ & \frac{1}{66} = \frac{5}{6} \left[ (x - x^{2} \frac{5}{4}) + 3x^{2} + 2x + 1 \right] dx. & 10. \int_{1}^{4} \frac{1 + x}{\sqrt{x}} dx. & 19. \int_{-1}^{1} (x - 3|x|) dx. \\ & x = \frac{1}{6} \left[ x + e^{x} + e^{x} + e^{x} + e^{x} \right] \text{ source}} & = 1 \text{ source} & = 1 \text{ source} \\ & 3. \int_{0}^{2} (x - 1) (x^{2} + 1) dx. & 11. \int_{0}^{1} \frac{1}{\sqrt{x}} \frac{1}{\sqrt{x}} + \frac{3}{\sqrt{x}} dx. & 20. \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \csc^{2} \theta d\theta. \\ & = 1 \text{ source} & \frac{1}{8} = \frac{9}{6} \left[ x - e^{x} \frac{5}{8} + e^{x} \frac{5}{8} - e^{x} \frac{5}{4} \right] \text{ source}} & 12. \int_{0}^{1} (\sqrt{x} \frac{1}{8} + \sqrt[3]{x}) dx. & 21. \int_{0}^{\frac{\pi}{4}} \frac{1 - \cos^{2} \theta}{\cos^{2} \theta} d\theta. \\ & = 1 \text{ source} & \frac{1}{8} = \frac{9}{6} \left[ x + e^{x} + e^{x} \frac{5}{8} + e^{x} \frac{1}{4} \right] \text{ source}} & 13. \int_{1}^{2} \left( x + \frac{1}{x} \right)^{2} dx. & = 1 \text{ source} \\ & \frac{1}{8} = \frac{9}{6} \left[ x + e^{x} + e^{x} \frac{5}{8} + e^{x} \frac{1}{4} \right] \text{ source}} & 14. \int_{1}^{2} \left( x + \frac{1}{x} \right)^{3} dx. & 23. \int_{0}^{\frac{\pi}{4}} \frac{\sin^{2} \theta}{\cos^{2} \theta} d\theta. \\ & 5. \int_{0}^{1} \left( 1 + x^{2} \right)^{3} dx. & = 1 \text{ source} \\ & \frac{\pi}{8} = \frac{9}{6} \left[ x + e^{x} + e^{x} \frac{5}{8} + e^{x} \frac{1}{4} \right] \text{ source}} & 14. \int_{1}^{2} \left( \sqrt{x} + \frac{1}{\sqrt{x}} \right)^{3} dx. & = 1 \text{ source} \\ & e^{-x + 4} = \frac{1}{6} \left[ x + \frac{1}{8} + \frac{1}{8} \right] \text{ source}} & 15. \int_{1}^{2} \left( \sqrt{x} + \frac{1}{\sqrt{x}} \right)^{3} dx. & = 1 \text{ source} \\ & e^{-x + 4} = \frac{1}{6} \left[ \frac{1}{8} - \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right] \text{ source}} & 16. \int_{1}^{2} \left( \sqrt{x} + \frac{1}{\sqrt{x}} \right)^{3} dx. & = 1 \text{ source} \\ & \frac{1}{8} \frac{1}{\sqrt{x}} \left( \frac{5}{\sqrt{x}} + \frac{1}{8} + \frac{1}{8} \right) \text{ source}} & 16. \int_{0}^{2} \left( x + \frac{1}{\sqrt{x}} \right)^{3} dx. & = 1 \text{ source} \\ & \frac{1}{8} \frac{1}{\sqrt{x}} \left( \frac{5}{\sqrt{x}} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right] \text{ source} & 16. \int_{0}^{2} \left[ x - 1 \left| \frac{1}{\sqrt{x}} \right|^{3} dx. & = 1 \text{ source} \\ & \frac{1}{8} \frac{1}{\sqrt{x}} \left( \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right) \frac{1}{8} \left[ \frac{1}{\sqrt{x}} \left( \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right) \right] \text{ source} & 25. \int_{0}^{\pi} \left[ \sin \theta - \sin \theta + \sin$$

**Solution.** 13.18

$$\int_{0}^{1} \left| x - \frac{1}{2} \right| dx = \int_{0}^{\frac{1}{2}} \left| x - \frac{1}{2} \right| dx + \int_{\frac{1}{2}}^{1} \left| x - \frac{1}{2} \right| dx \qquad \left| \begin{array}{c} \left| x - \frac{1}{2} \right| = \frac{1}{2} - x \text{ when } x \le \frac{1}{2} \\ x - \frac{1}{2} \right| = x - \frac{1}{2} \text{ when } x \ge \frac{1}{2} \\ \end{array} \right|$$

$$= \int_{0}^{\frac{1}{2}} \left( \frac{1}{2} - x \right) dx + \int_{\frac{1}{2}}^{1} \left( x - \frac{1}{2} \right) dx \\ = \left[ -\frac{x^{2}}{2} + \frac{x}{2} \right]_{0}^{\frac{1}{2}} + \left[ \frac{x^{2}}{2} - \frac{x}{2} \right]_{\frac{1}{2}}^{1} \\ = \left( -\frac{1}{8} + \frac{1}{4} \right) + \left( \frac{1}{2} - \frac{1}{2} - \left( \frac{1}{8} - \frac{1}{4} \right) \right) \\ = \frac{1}{4}$$

Problem 14. Evaluate the definite integral. The answer key has not been proofread, use with caution.

$$I. \int_{e}^{e^{3}} \frac{dx}{x\sqrt[3]{\ln x}}.$$

$$(1 - \frac{e^{3}}{2})\frac{\delta}{2} 2xaan$$

$$(1 - \frac{e^{3}}{2})\frac{\delta}{2} 2xaan$$

$$I. \int_{0}^{1} xe^{-x^{2}} dx.$$

$$\frac{1}{1 - \frac{e^{-x^{2}}}{2x^{2} + 1}} dx.$$

$$(1 + 2)u_{1} 2xaan$$

$$I. \int_{0}^{1} \frac{e^{x} + 1}{e^{x} + x} dx.$$

$$Eut \frac{1}{1 - 2x^{2}} dx.$$

$$Eut \frac{1}{1 - 2x^{2}} dx.$$

$$(\frac{1}{1 + 2})u_{1} 2xaan$$

$$(\frac{1}{1 + 2})u_{2} 2xaan$$

$$(\frac{1}{1 + 2})u_{1} 2xaan$$

$$(\frac{1}{1 + 2})u_{2} 2xaan$$

Solution. 14.4

$$\int_{1}^{2} \frac{x}{2x^{2}+1} dx = \int_{x=1}^{x=2} \frac{\frac{1}{4}d(2x^{2})}{2x^{2}+1} = \frac{1}{4} \int_{x=1}^{x=2} \frac{d(2x^{2}+1)}{2x^{2}+1} = \frac{1}{4}$$

**Problem 15.** Differentiate f(x) using the Fundamental Theorem of Calculus part 1.

 $^{2}(^{h}x + 2) -$ 

3. 
$$f(x) = \int_{0}^{x^{2}} t^{2} dt.$$
  
4.  $f(x) = \int_{\ln x}^{e^{x}} t^{3} dt.$   
5.  $g(x) = \int_{0}^{x^{3}} \cos^{2} t dt$ 

**Solution.** 15.2 We recall that the Fundamental Theorem of Calculus part 1 states that  $\frac{d}{dx} \left( \int_a^x h(t) dt \right) = h(x)$  where a is a constant. We can rewrite the integral so it has x as the upper limit:

$$f(x) = \int_{x}^{1} (2+1^{4})^{5} dt = -\int_{1}^{x} (2+1^{4})^{5} dt \quad .$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( -\int_1^x (2+t^4)^5 \mathrm{d}t \right) = -\frac{\mathrm{d}}{\mathrm{d}x} \left( \int_1^x (2+t^4)^5 \mathrm{d}t \right) \stackrel{\mathrm{FTC}\,\mathrm{part}\,1}{=} -(2+x^4)^5 \quad .$$

Solution. 15.4

$$f'(x) = \frac{d}{dx} \left( \int_{\ln x}^{e^x} t^3 dt \right)$$
$$= \frac{d}{dx} \left( \int_{\ln x}^{0} t^3 dt + \int_{0}^{e^x} t^3 dt \right)$$
$$= \frac{d}{dx} \left( -\int_{0}^{\ln x} t^3 dt + \int_{0}^{e^x} t^3 dt \right).$$

The Fundamental Theorem of Calculus part I states that for an arbitrary constant a,  $\frac{d}{du}\left(\int_{a}^{u}g(t)dt\right) = g(u)$  (for a continuous g). We use this two compute the two derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( \int_{0}^{\ln x} t^{3} \mathrm{d}t \right) = \frac{\mathrm{d}}{\mathrm{d}x} \left( \int_{0}^{u} t^{3} \mathrm{d}t \right) \qquad \left| \begin{array}{c} \operatorname{Set} u = \ln x \\ = u^{3} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \\ = \frac{(\ln x)^{3}}{x} \\ \frac{\mathrm{d}}{\mathrm{d}x} \left( \int_{0}^{e^{x}} t^{3} \mathrm{d}t \right) = \frac{\mathrm{d}}{\mathrm{d}x} \left( \int_{0}^{w} t^{3} \mathrm{d}t \right) \qquad \left| \begin{array}{c} \operatorname{Set} w = e^{x} \\ = w^{3} \cdot \frac{\mathrm{d}w}{\mathrm{d}x} \\ = e^{3x} e^{x} = e^{4x} \end{array} \right|$$

Finally, we combine the above computations to a single answer.

$$f'(x) = e^{4x} - \frac{(\ln x)^3}{x}.$$

**Problem 16.** 1. Find the area of the region bounded by the curves  $y = 2x^2$  and  $y = 4 + x^2$ .

2. Find the area of the region bounded by the curves  $x = 4 - y^2$  and y = 2 - x.

 $\frac{x}{\varepsilon^{(x u l)}} - x \overline{\psi}^{2} = (x) f : \text{Mesure}$ 

 $\binom{x}{2}$  z<sup>SOD</sup> z<sup>XE</sup> : *AMSUD* 

Solution. 16.2.  $x = 4 - y^2$  is a parabola (here we consider x as a function of y). y = -x + 2 implies that x = 2 - y and so the two curves intersect when

$$\begin{array}{rcrcrcrc} 4 - y^2 &=& 2 - y \\ -y^2 + y + 2 &=& 0 \\ -(y+1)(y-2) &=& 0 \\ y &=& -1 \text{ or } 2 \end{array}$$

As x = 2 - y, this implies that x = 0 when y = 2 and x = 3 when y = -1, or in other words the points of intersection are (0, 2) and (3, -1). Therefore we the region is the one plotted below. Integrating with respect to y, we get that the area is

$$A = \int_{-1}^{2} |4 - x^2 - (-x + 2)| \, dy = \int_{-1}^{2} (-y^2 + y + 2) \, dy$$
  
=  $\left[ -\frac{y^3}{3} + \frac{y^2}{2} + 2y \right]_{-1}^{2} = -\frac{8}{3} + 2 + 4 - \left( -\frac{(-1)^3}{3} + \frac{(-1)^2}{2} - 2 \right)$   
=  $\frac{9}{2}$ .



Problem 17.

1. Consider the region bounded by the curves  $y = 2x^2 - x + 1$  and  $y = x^2 + 1$ . What is the volume of the solid obtained by rotating this region about the line x = 0?

·и G :ләмѕир

2. Consider the region bounded by the curves  $y = 1 - x^2$  and y = 0. What is the volume of the solid obtained by rotating this region about the line y = 0?

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3. Consider the region bounded by the curves  $y = x^2$  and  $x = y^2$ . What is the volume of the solid obtained by rotating this region about the line x = 2?

<u>08</u> <u>₩18</u> :Jamsub **Solution.** 17.1 First, plot  $y = 2x^2 - x + 1$  and  $y = x^2 + 1$ .



Therefore the two points of intersection have x-coordinates between x = 0 and x = 1. Therefore we need to integrate the volumes of washers with inner radii  $r_{inner} = 2x^2 - x + 1$ , outer radii  $r_{outer} = x^2 + 1$  and infinitesimal heights dx. The volume of an individual infinitesimal washer is then  $\pi(r_{outer}^2 - r_{inner}^2)dx$ 

$$V = \int_0^1 \pi \left( \left( x^2 + 1 \right)^2 - \left( 2x^2 - x + 1 \right)^2 \right) dx$$
  
=  $\pi \int_0^1 \left( -3x^4 + 4x^3 - 3x^2 + 2x \right) dx$   
=  $\pi \left[ -\frac{3}{5}x^5 + x^4 - x^3 + x^2 \right]_0^1$   
=  $\frac{2}{5}\pi$ .