The real numbers: the only complete ordered field.

Theorem:

(a) [existence] The **Real Numbers form a complete ordered field**.

(b) [uniqueness] Up to isomorphism, there is only one complete ordered field.

This document is devoted to a (partial) proof of the above statement.

Remark: in order to do calculus, we need to use the **completeness property of the real numbers**.

This property is used in the text, but not proved. The completeness property of R is statement (7) and (8) below.

A full proof is a bit beyond the scope of this course; but what follows here can lead to such a proof.

Definitions.

- 1. algebraically, a **<u>field</u> F** is a set with two operations, addition (+) and multiplication ([']) satisfying:
 - a. Closure. F is closed under addition and multiplication.
 - b. Associativity: addition and multiplication are associative.
 - c. Identity:
 - i. There is an identity element (0) such that $\forall x \in \mathbf{F}$, 0+x = x+0 = x.
 - ii. There is an identity element (1) such that $\forall x \in F$, 1x = x1 = x.

d. Inverses:

- i. (Addititive inverses) $\forall x \in \mathbf{F}, \exists y \in \mathbf{F}$, such that x+y = y+x = 0. We usually write y = -x.
- ii. (Multiplicative inverses) $\forall x \in F$, $\underline{if \ x \neq 0}$, then $\exists y \in F$, such that xy = yx = 1. We usually write $y = x^{-1}$ or y = 1/x.
- e. **Distributivity** of multiplication over addition: $\forall x \in \mathbf{F}, \forall y \in \mathbf{F}, \forall z \in \mathbf{F}, x(y+z) = xz + yz.$

[Note: this is the only law that connects addition with multiplication !]

- f. Commutativity: $\forall x \in \mathbf{F}, \forall y \in \mathbf{F}$, then:
 - i. x + y = y + x.
 - ii. xy = yx.
- g. $1 \neq 0$. This implies that there is more than one element in the field **F**.

2. A field \mathbf{F} is called an <u>ordered field</u> iff

- a. [Trichotomy] **F** can be separated into three disjoint subsets: $\mathbf{F} = \mathbf{P} \cup \{0\} \cup \mathbf{N}$, (that is, positive numbers, zero, and negative numbers) with
- b. $\forall x \in \mathbf{P}, \forall y \in \mathbf{P}, x+y \in \mathbf{P}$
- c. $\forall x \in \mathbf{F}$, if $x \neq 0$ then $x^2 \in \mathbf{P}$

Then, we define $\mathbf{x} < \mathbf{y}$ iff $y - x \in \mathbf{P}$. We define $\mathbf{y} > \mathbf{x}$ if x < y.

The following properties of "<" can be easily proved.

d. If x < y and y < z, then x < z

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- e. If 0 < x, and if y < z, then xy < xz.
- f. If x < 0, and if y < z, then xz < xy.
- 3. Lemma: The rational numbers **Q** form an ordered field. Proof: (omitted).
- 4. Definition: The real numbers R consist of all numbers that can be represented by either an integer, a rational number (i.e. a repeating decimal), or a decimal with a non-repeating fractional part.
- 5. Remark:

But we must make the identification that 0.99999... (repeating) = 1.

Define addition and multiplication using elementary math operations. (It isn't totally easy to multiply two decimal numbers whose fractional parts don't terminate, but you can figure out how to do this).

- 6. Theorem: the infinitely repeating decimal sequence 0.9999999..... is equal to 1.
 - a. Proof #1, by division.

Divide 1 by 9 using long division. the result is 0.1111111....

Multiply the result by 9. the result is 9(1/9) = 1. But it is also .9999999.... so the two are equal.

b. Proof #1, by "intuition".

.9 < .99 < .999 < .9999, and so on. Each terminating decimal listed is < 1 or equal to 1. But consider 1 - .9999..... (the infinite repeating decimal). This number cannot be less than 1. For if it were, then there would be some nonzero number t, with 1 - .9999.... = t, and t > 0. But the decimal representation for t could have no non-zero digits. Therefore 1 = .999999.....

- c. [Note: neither of the above proofs is completely convincing, because we haven't given a completely bullet-proof definition of the Real Numbers. However, our definition will do for the purposes of this course.]
- Definition [completeness]: an ordered field F is called a complete ordered field iff: If J is any non-empty subset of F with an upper bound, then J has a least upper bound in F.
 - a. An **upper bound** b of J is any element b of F such that $\forall x \in J, x \leq b$. (Note that b could be an element of J, or b could not be an element of J).
 - b. A **least upper bound** b of J is an upper bound b of J, such that if c is any other upper bound of J, then $b \le c$.
- 8. Proposition: The real numbers R form a complete ordered field.
- 9. Proposition: Suppose S is any complete ordered field. Then S may be put into 1-1 correspondence with the real numbers R by a map f:R → S, such that:
 - a. f(0) = 0. f(1) = 1.
 - b. f(x+y) = f(x) + f(y). [Here, addition on the left side is within R; addition on the right side is within S. That is, the map f preserves addition.]

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- c. $f(xy) = f(x)^{-} f(y)$. [Here, multiplication on the left side is within R; multiplication on the right side is within S. That is, the map f preserves multiplication.]
- d. If x,y are elements of R, and if x < y in R, then f(x) < f(y) in S. That is, the map f preserves "less than".
- e. f(a/b) = f(a)/f(b), and so if a and be are rational numbers, you can see that the map f preserves rational numbers.
- f. Every real is a least upper bound of the set of terminating decimals less than it. Example:

 π = the least upper bound of {3,3.1,3.14,3.141,3.1415,3.14159,...3.14159265,...} $\sqrt{2}$ = the least upper bound of {1,1.4,1.41,1.414,1.4142, ...}

So each real number is the least upper bound of a set of rational numbers.

But because f preserves multiplication, f(a/b) = f(a)/f(b), and so f preserves rational numbers.

g. If J is a nonempty subset of R, and b is the least upper bound of J, then f(b) is the least upper bound of the set f(J) in S.

Proof of (g): omitted. Sketch of proof: you need to look at rational numbers in R, and rational numbers in S.

Then it has been demonstrated that the real numbers R form the only complete ordered field.