MATH 140 Spring 2015 Final exam practice problems - Answer Key

1. Find the following limits, or show that they do not exist:

   (a) $\lim_{y\to 2} \frac{y^2-4}{y^2-y-2} = \frac{4}{3}$

   (b) $\lim_{x\to 0^+} \frac{x}{\sqrt{1+x}-1} = 2$

   (c) $\lim_{x\to 1^-} \frac{x}{\ln x} = -\infty$

   (d) $\lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = e^x$

   (e) $\lim_{x\to -\infty} \frac{5x^3+x-1}{2x^3-7} = \frac{5}{2}$

   (f) $\lim_{z\to 1^+} \frac{z-3}{z-1} = -\infty$

   (g) $\lim_{x\to\infty} \frac{\sqrt{9x^3-2}}{x^2+4} = 0$

   (h) $\lim_{h\to 0} \frac{2(x+h)^3-2x^3}{h} = 6x^2$

2. (a) $\frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$.

   (b) $\frac{2}{x}+1$.

   (c) $6\sin^2(2\theta)\cos(\theta)$.

   (d) $2x+\sec^2 x$.

   (e) $-(2+x^4)^5$.

   (f) $\cos^2(x^3)\cdot 3x^2$.

   (g) $\frac{dy}{dx} = \frac{-4x-1-y}{x}$.

   (h) $\frac{dy}{dx} = \frac{-\sin y - y\cos x}{x\cos y+\sin x}$.

3. (a) $\frac{x^3}{3}+3\ln|x|+C$.

   (b) $\ln|\sin x|+C$.

   (c) $x+\frac{1}{2}e^{2x}+C$.

   (d) $\frac{2}{3}(\ln x)^{3/2}+C$.

4. $3x+\ln|x|-1$.

5. $\frac{x^2}{2}+\cos x+6$.

6. Let $f(x) = x^4 - 8x^2 + 8$.
   (a) Increasing on $(-2, 0)$ and $(2, \infty)$, and decreasing on $(-\infty, -2)$ and $(0, 2)$.
   (b) Concave up on $(-\infty, -2/\sqrt{3})$ and $(2/\sqrt{3}), \infty)$. Concave down on $(-2/\sqrt{3}, 2/\sqrt{3})$.
   (c) Check your graph with www.desmos.com.
7. Let $f(x) = x^5 - 15x^3$.
   (a) Increasing on $(-\infty, -3)$ and $(3, \infty)$ and decreasing on $(-3, 0)$ and $(0, 3)$ (f is briefly flat at $x = 0$).
   (b) Concave up on $(-3/\sqrt{2}, 0)$ and $(3/\sqrt{2}, \infty)$, and concave down on $(-\infty, -3/\sqrt{2}$ and $(0, 3/\sqrt{2})$.
   (c) Check your graph with www.desmos.com.
8. Let $f(x) = \frac{x+2.5}{x^2-4}$.
   (a) Horizontal: $y = 0$. Vertical: $x = 2$ and $x = -2$.
   (b) Increasing on $(-4, -2)$ and $(-2, -1)$. Decreasing on $(-\infty, -4)$, $(-1, 2)$, and $(2, \infty)$. (The derivative is undefined at $x = 2$ and $x = -2$.)
   (c) Check your graph with www.desmos.com.
9. Let $f(x) = \frac{2x^3+1}{x^3-1}$.
   (a) Horizontal: $y = 2$. Vertical: $x = 1$.
   (b) Decreasing on $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. (The derivative is 0 at $x = 0$ and undefined at $x = 1$.)
   (c) Check your graph with www.desmos.com
10. $(1.001)^9 \approx 1.009$
11. $e^{0.2} \approx 1.2$.
12. $9 + \sqrt{17}$
13. 83/65
14. $(3, 2)$
15. 12 cubic feet
16. Let $f(x) = x^3 + 4x - 7$.
   (a) Use the Intermediate Value Theorem to show that $f(x)$ has at least one root.
   First of all, f is a polynomial, and so it is continuous. **It is really important to say this, because otherwise the Intermediate Value Theorem does not apply!** We note that $f(0) = -7$ and $f(2) = 9$, and so f has both negative and positive outputs. Then, since $f(0) < 0 < f(2)$, the Intermediate Value Theorem says that for some c such that $0 < c < 2$, we have $f(c) = 0$. In other words, $f(x)$ has a root on the interval $(0, 2)$.

(b) Use the Mean Value Theorem or Rolle's Theorem to show that $f(x)$ has at most one root.

Since f is a polynomial, it is continuous and differentiable everywhere, and so Rolle's Theorem applies. Suppose that $f(x)$ has two or more roots, and pick two of them, say $x = a$ and $x = b$. In other words, $f(a) = f(b) = 0$. Then by Rolle's Theorem, $f'(c) = 0$ for some value of c. On the other hand, we note that $f'(x) = 3x^2 + 4$, and therefore, $f'(x) \geq 4$ for all x. Then it cannot happen that $f'(c) = 0$, and having arrived at a contradiction, that means that our assumption was wrong. So $f(x)$ does not have two or more roots; in other words, it has at most 1 root.

17. Let $g(x) = e^x + x + 2$.

(a) Use the Intermediate Value Theorem to show that $g(x)$ has at least one root.

The function $g(x)$ is the sum of e^x and $x + 2$, both of which we know to be continuous, and thus g is itself continuous. So we may apply the Intermediate Value Theorem. We note that $g(-10) = e^{-10} - 10 + 2 = e^{-10} - 8$, and since $e^{-10} = 1/e^10$, that part is a very small number, and so $e^{-10} - 8$ is negative. We also note that $g(0) = e^0 + 0 + 2 = 1 + 0 + 2 = 3$. Therefore, $g(-10) < 0 < g(0)$. Thus, the Intermediate Value Theorem says that for some c such that $-10 < c < 0$, we have $g(c) = 0$. In other words, $g(x)$ has a root on the interval $(-10, 0)$.

(b) Use the Mean Value Theorem or Rolle's Theorem to show that $g(x)$ has at most one root.

Again, g is the sum of e^x and $x+2$, both of which we know to be continuous and differentiable, and so Rolle's Theorem applies. Suppose that $g(x)$ has two or more roots, and pick two of them, say $x = a$ and $x = b$. In other words, $g(a) = g(b) = 0$. Then by Rolle's Theorem, $g'(c) = 0$ for some value of c. On the other hand, we note that $g'(x) = e^x + 1$, and since e^x is always positive, $g'(x) \geq 1$ for all x. Then it cannot happen that $g'(c) = 0$, and having arrived at a contradiction, that means that our assumption was wrong. So $g(x)$ does not have two or more roots; in other words, it has at most 1 root.

18. $dr/dt = -\sqrt{10}/8\sqrt{\pi}$

19. $dy/dt = -9/10$.

20. $32/3$

21. $e^2 - \frac{11}{3}$

22. $16\pi/15$

23. $3\pi/10$