MATH 140 Spring 2015 Final exam practice problems - Solutions

1. Find the following limits, or show that they do not exist:

   (a) $\lim_{y\to 2} \frac{y^2-4}{y^2-y-2}$
   Substituting $y=2$ gives us $0/0$, so we look for a way to simplify:
   $$\frac{y^2-4}{y^2-y-2} = \frac{(y-2)(y+2)}{(y-2)(y+1)} = \frac{y+2}{y+1}.$$
   Now substituting $y=2$ yields $4/3$.

   (b) $\lim_{x\to 0^+} \frac{x}{\sqrt{1+x}-1}$
   Substituting $x=0$ gives $0/0$. To simplify, we multiply top and bottom by $\sqrt{1+x}+1$, giving us:
   $$\frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} = \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x})^2-1} = \frac{x(\sqrt{1+x}+1)}{x} = \sqrt{1+x}+1.$$
   Now substituting $x=0$ gives us 2.

   (c) $\lim_{x\to 1^-} \frac{x}{\ln x}$
   Substituting $x=1$ yields $1/\ln(1)$, and since $\ln(1)=0$, we get $1/0$. Since we have nonzero / zero, that indicates an infinite limit. Since x is approaching 1 from the left, we consider x-values a little less than 1; in that case, $\ln x$ is a little less than 0. In other words, $\ln x$ is a small negative number, and so we have 1 / (small negative), which gives a large negative. Thus, we see that we get a limit of $-\infty$.

   (d) $\lim_{h\to 0} \frac{e^{x+h}-e^x}{h}$
   Substituting $h=0$ gives us $0/0$. This one is difficult to solve directly, but there is a nice trick. The form of the limit looks familiar; remember that the definition of $f'(x)$ is $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Therefore, the given limit stands for the derivative of e^x, which is e^x.

   (e) $\lim_{x\to -\infty} \frac{5x^3+x-1}{2x^3-7}$
   "Substituting" $x=-\infty$ gives us $-\infty/-\infty$. Just like $0/0$, this is not really an answer; we have to find a way to simplify the function. If we divide top and bottom by x^3, we get $\frac{5+\frac{1}{x^2}-\frac{1}{x^3}}{2-\frac{7}{x^3}}$. Now as $x\to\infty$, the terms with a constant on top and a power of x on bottom shrink to 0, and we are left with $5/2$.

   (f) $\lim_{z\to 1^+} \frac{z-3}{z-1}$
   Substituting $z=1$ gives us $-2/0$, which indicates an infinite limit. To see whether we get ∞ or $-\infty$, we consider a z-value a little larger than 1 (since we are approaching 1 from the right). In that case, the top is a little larger than -2, and the bottom is a little larger than 0. Since we get a negative over a positive, that means we are getting negative outputs as z approaches 1 from the right, and thus the limit is $-\infty$.

(g) $\lim_{x\to\infty} \dfrac{\sqrt{9x^3-2}}{x^2+4}$

"Substituting" $x=\infty$ gives ∞/∞, and we need to simplify the function. Let us divide the top and bottom by x^2. On bottom, we will get $1+\frac{4}{x^2}$. On top we need to be careful; we cannot just divide the terms inside the square root by x^2. We note that $x^2=\sqrt{x^4}$, and so

$$\frac{\sqrt{9x^3-2}}{x^2} = \frac{\sqrt{9x^3-2}}{\sqrt{x^4}} = \sqrt{\frac{9x^3-2}{x^4}} = \sqrt{\frac{9}{x}-\frac{2}{x^4}}.$$

Thus, after dividing the top and bottom of our original function by x^2, we get

$$\frac{\sqrt{\frac{9}{x}-\frac{2}{x^4}}}{1+\frac{4}{x^2}}.$$

Now when we substitute $x=\infty$, we get $0/1$, which is 0.

(h) $\lim_{h\to 0} \dfrac{2(x+h)^3-2x^3}{h}$

Substituting $h=0$ gives $0/0$. It is possible to do this directly, by expanding $(x+h)^3$, but that is a little tricky. (In particular, $(x+h)^3 = x^3+3x^2h+3xh^2+h^3$.) A better approach is to notice, as in part (d), that this limit looks like the definition of a derivative. Indeed, this limit represents the derivative of $2x^3$, and so the value of the limit is $6x^2$.

2. Find the derivative of the following functions.

(a) $e^{\sqrt{x^2+1}}$

$$\frac{d}{dx}(e^{\sqrt{x^2+1}}) = e^{\sqrt{x^2+1}}\cdot\frac{d}{dx}(\sqrt{x^2+1}) = e^{\sqrt{x^2+1}}\cdot\frac{1}{2\sqrt{x^2+1}}\cdot\frac{d}{dx}(x^2+1) = e^{\sqrt{x^2+1}}\cdot\frac{1}{2\sqrt{x^2+1}}\cdot 2x = \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}.$$

(b) $\ln(x^2e^x)$

Solution 1: For any quantities u and v, $\ln(uv)=\ln(u)+\ln(v)$. Therefore, $\ln(x^2e^x) = \ln(x^2)+\ln(e^x)$. This can be simplified further using properties of logarithms; $\ln(x^2)=2\ln x$ and $\ln(e^x)=x$. Therefore,

$$\frac{d}{dx}(\ln(x^2e^x)) = \frac{d}{dx}(2\ln x + x) = \frac{2}{x}+1.$$

Solution 2: We can also do this using the chain rule and product rule:

$$\frac{d}{dx}(\ln(x^2e^x)) = \frac{1}{x^2e^x}\cdot\frac{d}{dx}(x^2e^x) = \frac{2xe^x+x^2e^x}{x^2e^x} = \frac{2}{x}+1.$$

(c) $\sin^3(2\theta)$

Remember that $\sin^3(2\theta)$ is a shorthand for $(\sin(2\theta))^3$. Then, using the chain rule twice, we get

$$\frac{d}{d\theta}((\sin(2\theta))^3) = 3(\sin(2\theta))^2\cdot\frac{d}{d\theta}(\sin(2\theta)) = 3(\sin(2\theta))^2\cdot\cos(2\theta)\cdot\frac{d}{d\theta}(2\theta) = 3(\sin(2\theta))^2\cdot\cos(2\theta)\cdot 2 = 6\sin^2(2\theta)\cos(\theta).$$

(d) $x\left(x + \dfrac{\tan x}{x}\right)$

You could use the product rule and then the quotient rule, but it is much easier if you first simplify the function by multiplying the x in, giving you $x^2 + \tan x$. Therefore,

$$\frac{d}{dx}\left(x\left(x + \frac{\tan x}{x}\right)\right) = \frac{d}{dx}(x^2 + \tan x) = 2x + \sec^2 x.$$

(e) $f(x) = \displaystyle\int_x^1 (2+t^4)^5\, dt$

The Fundamental Theorem of Calculus says that (for sufficiently "nice" functions g),

$$\frac{d}{dx}\left(\int_a^x g(t)\, dt\right) = g(x).$$

Since switching the upper and lower bound of an integral just changes the sign, we get

$$\frac{d}{dx}\left(\int_x^1 (2+t^4)^5\, dt\right) = -\frac{d}{dx}\left(\int_1^x (2+t^4)^5\, dt\right) = -(2+x^4)^5.$$

(f) $g(x) = \displaystyle\int_5^{x^3} \cos^2 t\, dt$

In order to use the Fundamental Theorem of Calculus, we need the upper bound to be a single variable, not an expression like x^3. If we make the substitution $u = x^3$, then

$$\frac{d}{du}\left(\int_5^{x^3} \cos^2 t\, dt\right) = \frac{d}{du}\left(\int_5^{u} \cos^2 t\, dt\right) = \cos^2(u).$$

But we want dg/dx, not dg/du. The chain rule tells us that $dg/dx = dg/du \cdot du/dx$, and so

$$\frac{dg}{dx} = \cos^2(u)\cdot 3x^2 = \cos^2(x^3)\cdot 3x^2.$$

(g) Find dy/dx if $2x^2 + x + xy = 1$.

We start by taking d/dx of both sides, remembering to apply the product rule when we take d/dx of xy:

$$4x + 1 + (1\cdot y + x\cdot \frac{dy}{dx}) = 0.$$

Now we solve for dy/dx by subtracting the first three terms from both sides and then dividing by x:

$$\frac{dy}{dx} = \frac{-4x-1-y}{x}.$$

(h) Find y' if $x\sin y + y\sin x = 4$.

We start by taking d/dx of both sides, applying the product rule to both terms on the left:

$$(1\cdot \sin y + x\cdot \cos y \cdot \frac{dy}{dx}) + (\frac{dy}{dx}\cdot \sin x + y\cdot \cos x) = 0.$$

Now we move the terms without dy/dx over to the right:

$$x\cos y\frac{dy}{dx}+\frac{dy}{dx}\sin x=-\sin y-y\cos x.$$

Finally, we factor dy/dx out of both terms on the left and divide by the coefficient that remains:

$$\frac{dy}{dx}=\frac{-\sin y-y\cos x}{x\cos y+\sin x}.$$

3. Evaluate the following integrals.

(a) $\displaystyle\int\frac{x^4+3x}{x^2}\,dx$

Just like derivatives, integrals don't "split up" over division, so you cannot just deal with the top and the bottom separately. Instead, we start by simplifying the function on the inside:

$$\frac{x^4+3x}{x^2}=\frac{x^4}{x^2}+\frac{3x}{x^2}=x^2+\frac{3}{x}.$$

Therefore,

$$\int\frac{x^4+3x}{x^2}\,dx=\int(x^2+\frac{3}{x})\,dx=\frac{x^3}{3}+3\ln|x|+C.$$

(b) $\displaystyle\int\frac{\cos x}{\sin x}\,dx$

Again, we cannot just take the antiderivative of each part and divide them. Instead, we make the substitution $u=\sin x$, which also gives us $du=\cos x\,dx$. Then our original integral translates to $\int\frac{1}{u}\,du$. The general antiderivative of $1/u$ is $\ln|u|+C$, and since $u=\sin x$, we get $\ln|\sin x|+C$.

(c) $\displaystyle\int(1+e^x)(1-e^x)\,dx$

For finding integrals, we only have two basic tools at our disposal: algebraic manipulation, and substitution. It is often a good idea to try algebraic manipulation first, and here we can just multiply the two binomials:

$$(1+e^x)(1-e^x)=1+e^x-e^x-(e^x)^2=1-e^{2x}.$$

Thus, we want to find

$$\int(1-e^{2x})\,dx.$$

Integrals do split up over addition and subtraction, so this is equal to

$$\int 1\,dx-\int e^{2x}\,dx.$$

The first integral is just x (and we will omit the "+C" here and just put a single one on at the end). To integrate e^{2x} requires a minor substitution; we set $u=2x$, in which case $du=2dx$ and thus $dx=\frac{1}{2}du$. Therefore,

$$\int e^{2x}\,dx=\int\frac{1}{2}e^u\,du=\frac{1}{2}e^u+C=\frac{1}{2}e^{2x}+C.$$

Putting it all together yields

$$x + \frac{1}{2}e^{2x} + C.$$

(d) $\displaystyle\int \frac{\sqrt{\ln x}}{x}\, dx$

There is no obvious algebraic manipulation, so we look for a substitution to try. When you are unsure, try picking the inside of a function for u. For example, here we have $\ln x$ all under a square root, so we try $u = \ln x$. This gives us $du = \frac{1}{x}\, dx$, and after translating x and dx we get

$$\int \sqrt{u}\, du.$$

(Try expanding out that integral using $u = \ln x$ and $du = \frac{1}{x}\, dx$ to convince yourself that it is the same as what we started with.) Now, $\sqrt{u} = u^{1/2}$, and therefore,

$$\int \sqrt{u}\, du = \int u^{1/2}\, du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}(\ln x)^{3/2} + C.$$

4. Find $f(x)$ if $f'(x) = 3 + \frac{1}{x}$ and $f(1) = 2$.

Since we are given $f'(x)$, we find $f(x)$ by integrating the given function. We get that

$$f(x) = \int f'(x)\, dx = \int (3 + \frac{1}{x})\, dx = 3x + \ln|x| + C.$$

Technically, what we have at the moment is a family of functions, one for each value that C can take. We want a single, specific function that satisfies $f(1) = 2$. Thus, we want

$$2 = f(1) = 3 + \ln|1| + C.$$

Since $\ln 1 = 0$, we get that $2 = 3 + C$, and so $C = -1$. Therefore, $f(x) = 3x + \ln|x| - 1$. We note that checking our answer is simple; it is true that $f'(x) = 3 + \frac{1}{x}$, and it is also true that $f(1) = 2$.

5. Find $g(x)$ if $g'(x) = x - \sin x$ and $g(0) = 7$.

Proceeding as the previous problem, we start with

$$g(x) = \int (x - \sin x)\, dx = \frac{x^2}{2} + \cos x + C.$$

We solve for C by setting 7 equal to $g(0)$:

$$7 = g(0) = 0 + \cos 0 + C = 1 + C.$$

Therefore, $C = 6$, and our answer is $g(x) = \frac{x^2}{2} + \cos x + 6$.

6. Let $f(x) = x^4 - 8x^2 + 8$.

(a) Find the intervals of increase and decrease.

We start by finding the critical points, which we get by setting $f'(x) = 0$ and solving. We get that $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$, and thus we get three critical points: $x = -2$, $x = 0$, and $x = 2$. So we get 4 ranges: $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$.

For each interval, we try a number in that range to see whether the derivative is positive or negative. For $x < -2$, we see that if we plug in -3, for example, we get the product of three negative numbers, so it is negative. For $-2 < x < 0$, if we try $x = -1$, then we get one positive number times two negative numbers, yielding a positive number. Repeating the process on the two remaining ranges shows that for $0 < x < 2$, we get a negative derivative, and for $x > 2$, we get a positive derivative. Thus, $f(x)$ is increasing on $(-2, 0)$ and $(2, \infty)$, and it is decreasing on $(-\infty, -2)$ and $(0, 2)$.

(b) Find the intervals where f is concave up, and where it is concave down.

We proceed similarly to the previous part, but with $f''(x)$ instead of $f'(x)$. Since $f'(x) = 4x^3 - 16x$, we see that $f''(x) = 12x^2 - 16$. Setting that to 0 and solving gives us that $x^2 = 16/12 = 4/3$, and thus $x = \pm 2/\sqrt{3}$. So we get three ranges: $(-\infty, -2/\sqrt{3})$, $(-2/\sqrt{3}, 2/\sqrt{3})$, and $(2/\sqrt{3}, \infty)$. Picking numbers in each range and putting them in to $f''(x)$ shows that $f''(x)$ is positive on $(-\infty, -2/\sqrt{3})$ and $(2/\sqrt{3}), \infty)$, and so f is concave up on that range. On $(-2/\sqrt{3}, 2/\sqrt{3})$, we get that $f''(x)$ is negative, and so f is concave down on that range.

(c) Sketch a graph that fits your information.

Check your sketch with a graphing tool such as the one at desmos.com.

7. Let $f(x) = x^5 - 15x^3$.

(a) Find the intervals of increase and decrease.

First we find the critical points. $f'(x) = 5x^4 - 45x^2 = 5x^2(x^2 - 9) = 5x^2(x+3)(x-3)$. Thus, $f'(x) = 0$ for $x = 0$, $x = -3$, and $x = 3$. So we get 4 ranges to test: $(-\infty, -3)$, $(-3, 0)$, $(0, 3)$, and $(3, \infty)$. Testing x-values in those ranges shows that $f'(x)$ is positive on $(-\infty, -3)$ and on $(3, \infty)$, and so f is increasing on those intervals. On the other hand, $f'(x)$ is negative on $(-3, 0)$ and $(0, 3)$, and so f is decreasing on $(-3, 3)$.

(b) Find the intervals where f is concave up, and where it is concave down.

We find that $f''(x) = 20x^3 - 90x = 10x(2x^2 - 9)$. Setting that to 0 and solving yields $x = 0$ or $2x^2 = 9$, which means $x = \pm\sqrt{9/2} = \pm 3/\sqrt{2}$. Then on $(-\infty, -3/\sqrt{2})$, we find $f''(x)$ is negative, and so f is concave down. On $(-3/\sqrt{2}, 0)$, we get that $f''(x)$ is positive, so f is concave up. On $(0, 3/\sqrt{2})$, $f''(x)$ is negative, so f is concave down, and on $(3/\sqrt{2}, \infty)$, $f''(x)$ is positive, and so f is concave up.

(c) Sketch a graph that fits your information.

Check your sketch with a graphing tool such as the one at desmos.com.

8. Let $f(x) = \frac{x+2.5}{x^2-4}$.

(a) Find the equations of any horizontal and vertical asymptotes.

To find horizontal asymptotes, we look at the limit of $f(x)$ as $x \to \infty$. Dividing top and bottom of $f(x)$ by x, we get that

$$f(x) = \frac{1 + \frac{2.5}{x}}{x - \frac{4}{x}},$$

And as $x \to \infty$, the top approaches 1 while the bottom continues to get larger and larger. Thus, the limit is 0, which means that the horizontal line $y = 0$ is an asymptote of $f(x)$. Furthermore, note that we get the same result as $x \to -\infty$, and so the graph should approach the line $y = 0$ both to the right and to the left.

To find vertical asymptotes, we look for x-values that make the bottom 0 while making the top nonzero. Setting the bottom to 0 and solving yields $x^2 = 4$, so $x = \pm 2$, which does not make the top 0. So $x = -2$ and $x = 2$ are both vertical asymptotes. In particular,

$$\lim_{x \to 2^+} \frac{x + 2.5}{x^2 - 4} = \infty,$$

because as x approaches 2 from the right, both the top and the bottom are positive. Using similar arguments, we can find that

$$\lim_{x \to 2^-} \frac{x + 2.5}{x^2 - 4} = -\infty,$$

$$\lim_{x \to -2^+} \frac{x + 2.5}{x^2 - 4} = -\infty,$$

$$\lim_{x \to -2^-} \frac{x + 2.5}{x^2 - 4} = \infty.$$

(b) Find the intervals of increase and decrease.

We take the derivative of $f(x)$ using the quotient rule and then simplify:

$$f'(x) = \frac{(x^2-4)(x+2.5)' - (x+2.5)(x^2-4)'}{(x^2-4)^2} = \frac{(x^2-4)(1) - (x+2.5)(2x)}{(x^2-4)^2} = \frac{-x^2 - 5x - 4}{(x^2-4)^2} = \frac{-(x+1)(x+4)}{(x^2-4)^2}$$

Now, to find the intervals of increase and decrease, we need to find where $f'(x)$ is positive or negative. We note that the bottom of $f'(x)$ is always positive (as long as $x \neq \pm 2$), and so the sign of $f'(x)$ is the same as the sign of the top. We see that the top, $-(x+1)(x+4)$, is 0 when $x = -1$ and when $x = -4$, and thus we get three ranges: $(-\infty, -4)$, $(-4, -1)$, and $(-1, \infty)$. Checking the sign of $-(x+1)(x+4)$ over those ranges, we see that $f'(x)$ is positive for x-values in $(-4, -1)$ and negative otherwise. Thus, f is increasing on $(-4, -1)$ and decreasing on $(-\infty, -4)$ and $(-1, \infty)$. To be really precise, we should exclude $x = 2$ and $x = -2$ from those ranges, since $f'(x)$ is undefined there. So f is increasing on $(-4, -2)$ and $(-2, -1)$ and decreasing on $(-\infty, -4)$, $(-1, 2)$, and $(2, \infty)$.

(c) Sketch a graph that fits your information.

Check your sketch with a graphing tool such as the one at desmos.com.

9. Let $f(x) = \frac{2x^3+1}{x^3-1}$.

(a) Find the equations of any horizontal and vertical asymptotes.

To find horizontal asymptotes, we look at the limit of $f(x)$ as $x \to \infty$. Dividing top and bottom of $f(x)$ by x^3, we get that

$$f(x) = \frac{2 + \frac{1}{x^3}}{1 - \frac{1}{x^3}},$$

And as $x \to \infty$, the top approaches 2 while the bottom approaches 1, and we get a limit of 2. We get the same result as $x \to -\infty$, and so the graph should approach the line $y = 2$ both to the right and to the left.

We find the vertical asymptotes by setting the bottom to 0 and solving. That gives us $x^3 = 1$, which has the single solution $x = 1$. Therefore, $x = 1$ is a vertical asymptote. Furthermore,

$$\lim_{x\to 1^+} \frac{2x^3+1}{x^3-1} = \infty,$$

since as x gets close to 1 from the right, we get a positive over a positive, yielding a positive answer. In a similar way, we find that

$$\lim_{x\to 1^-} \frac{2x^3+1}{x^3-1} = -\infty.$$

(b) Find the intervals of increase and decrease.

We start by taking the derivative, using the quotient rule:

$$f'(x) = \frac{(x^3-1)(2x^3+1)' - (2x^3+1)(x^3-1)'}{(x^3-1)^2} = \frac{(x^3-1)(6x^2) - (2x^3+1)(3x^2)}{(x^3-1)^2} = \frac{-9x^2}{(x^3-1)^2}.$$

We note that the only x-value that gives us $f'(x) = 0$ is $x = 0$, and that the derivative is undefined at $x = 1$. Other than those x-values, anything we substitute for x gives us a negative over a positive, and so f is decreasing everywhere, except that it is briefly flat at $x = 0$ and undefined at $x = 1$.

(c) Sketch a graph that fits your information.

Check your sketch with a graphing tool such as the one at desmos.com.

10. Use a linear approximation to estimate $(1.001)^9$. Be clear about which function you are using ($f(x)$) and the point where you are taking the linearization (the a in the textbook).

We start by trying to pick a function $f(x)$ that seems naturally connected with $(1.001)^9$. The easiest choice is $f(x) = x^9$. Next, I want to pick an x-value a that is close to 1.001 but much easier to deal with. Often you will accomplish this by rounding; here we will use $a = 1$. Now, the first thing I have to do is find the linearization of x^9 at $a = 1$, which is just the function corresponding to the tangent line at that point. The general formula for this linearization is:

$$L(x) = f(a) + f'(a) \cdot (x - a).$$

We have $f(a) = 1^9 = 1$, and $f'(x) = 9x^8$, so $f'(a) = 9(1)^8 = 9$. Therefore, we get

$$L(x) = 1 + 9(x - 1) = 9x - 8.$$

A good check on whether you've done things correctly is this: you should be able to simplify your $L(x)$ into the form $L(x) = mx + b$ as we have done here.

Finally, once we have the linearization, we can approximate $(1.001)^9$ (which is $f(1.001)$) by calculating $L(1.001)$:

$$L(1.001) = 9(1.001) - 8 = 1.009.$$

11. Use a linear approximation to estimate $e^{0.2}$. Be clear about which function you are using ($f(x)$) and the point where you are taking the linearization (the a in the textbook).

The natural function to pick is $f(x) = e^x$, and we round 0.2 to get $a = 0$. Then $f(a) = e^0 = 1$, and since $f'(x) = e^x$ as well, then $f'(a) = e^0 = 1$ too. Thus:

$$L(x) = 1 + 1(x - 0) = 1 + x.$$

Therefore,

$$e^{0.2} \approx L(0.2) = 1.2.$$

12. Approximate $\int_0^4 \sqrt{8x+1}$ using a Riemann sum with 4 terms ($n = 4$) and taking the sample point x_i^* in each subinterval to be the left end point of that subinterval. (That is, estimate the integral using 4 rectangles and left endpoints.) (**No credit** for solving the integral exactly using the Fundamental Theorem; we want an approximation.)

    We start by splitting the interval $[0, 4]$ (indicated by the limits of the integral) into 4 pieces (indicated by n), and so we get ranges $[0, 1]$, $[1, 2]$, $[2, 3]$, and $[3, 4]$. Since each range has width 1, we set $\Delta x = 1$. Now, to approximate this integral, we calculate

    $$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x,$$

    where $f(x) = \sqrt{8x+1}$, the function inside the integral. In this case, we are told to pick the left endpoints as our sample points, so we want

    $$f(0)\Delta x + f(1)\Delta x + f(2)\Delta x + f(3)\Delta x.$$

    This gives us

    $$(1)(1) + (3)(1) + \sqrt{17}(1) + (5)(1) = 9 + \sqrt{17}.$$

13. Approximate $\int_0^6 \frac{1}{x^2+1}$ using a Riemann sum with 3 terms ($n = 3$) and taking the sample point x_i^* in each subinterval to be the midpoint of that interval. (**No credit** for solving the integral exactly using the Fundamental Theorem; we want an approximation.)

    We start by splitting the interval $[0, 6]$ (indicated by the limits of the integral) into 3 pieces (indicated by n), and so we get ranges $[0, 2]$, $[2, 4]$, and $[4, 6]$. Since each range has width 2, we set $\Delta x = 2$. Now, to approximate this integral, we calculate

    $$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x,$$

    where $f(x) = \frac{1}{x^2+1}$, the function inside the integral. In this case, we are told to pick the midpoints as our sample points, so we want

    $$f(1)\Delta x + f(3)\Delta x + f(5)\Delta x.$$

    This gives us

    $$(1/2)(2) + (1/10)(2) + (1/26)(2) = 1 + \frac{1}{5} + \frac{1}{13} = \frac{65}{65} + \frac{13}{65} + \frac{5}{65} = \frac{83}{65}.$$

14. What is the point on the hyperbola $xy - y = 4$ that is closest to the point $(1, 0)$?

    That key word "closest" tells us that this is an optimization problem: we are trying to make distance as small as possible. If we have a point (x, y), its distance to $(1, 0)$ is $\sqrt{(x-1)^2 + (y-0)^2}$. When dealing with problems of this sort (minimizing distance), there is a common trick that is useful: we just focus on minimizing the function that is under the square root. So we want to make $(x-1)^2 + y^2$ as small as possible. Next, we need to transform this function into a single-variable function. Since our point must lie on the hyperbola $xy - y = 4$, that means that $(x-1)y = 4$, and thus $y = 4/(x-1)$. Substituting that into $(x-1)^2 + y^2$ gives us that the function to minimize is $f(x) = (x-1)^2 + \frac{16}{(x-1)^2}$. We take its derivative and get $f'(x) = 2(x-1) - \frac{32}{(x-1)^3}$. Setting that to 0 and solving gives us that $2(x-1)^4 = 32$, and thus $(x-1)^4 = 16$, which means that $(x-1) = 2$; thus $x = 3$. This is our only critical point, and we can check that $f'(x)$ is negative when $x < 3$ and positive when $x > 3$, so this really is our local minimum. Since $f(x)$ has only a single critical point, and it is a local minimum, it follows that it is the absolute minimum. Finally, to get the corresponding y-value, we just put our x value back into the equation $y = 4/(x-1)$ to get $y = 2$. Therefore, the point $(3, 2)$ is the point on this hyperbola that is closest to $(1, 0)$.

15. You want to build a rectangular box with a square base out of sheet metal. You are going to use 2 pieces of sheet metal for the bottom of the box to reinforce it, and only a single piece of sheet metal for all of the sides and the top. If you want to use no more than 36 sq. ft. of material, what is the largest possible volume you can enclose?

    We are deciding on the dimensions to use for the box: width, depth, and height. Since the base must be a square, we have that width and depth are equal. So let us use x for the width and depth and y for the height. The 4 sides of the box (left, right, front, and back) have dimensions x by y, and so they have area xy. Since there are 4 of them, they contribute $4xy$ to the total area of metal used. The top and bottom both have dimensions x by x, and so they have area x^2. However, we are going to use two pieces of sheet metal for the bottom, which means we need 3 pieces of this area, instead of 2. Therefore, the total material we are going to use is $4xy + 3x^2$. We are only allowed to use 36 sq. ft, so we have the constraint that $4xy + 3x^2 = 36$.

    Now, our goal is to maximize volume, which is the product of width, depth, and height. Thus we have $V = x^2 y$. In order to maximize this, we first need to put things in terms of a single variable. If we solve our constraint for y, we get that $4xy = 36 - 3x^2$, and thus $y = \frac{36-3x^2}{4x}$. Putting that into the volume formula yields

    $$V = x^2 y = x^2 \frac{36 - 3x^2}{4x} = x \frac{36 - 3x^2}{4} = \frac{36x - 3x^3}{4}.$$

    This is now a single variable function, so we can take its derivative to find the critical points:

    $$V' = \frac{36 - 9x^2}{4}.$$

    Setting that to 0 and solving gives us that $9x^2 = 36$, and so $x^2 = 4$ and thus $x = \pm 2$. Of course, dimensions must be positive, so $x = 2$. We can check that this really is a local maximum, because V' is positive when $x < 2$ and negative when $x > 2$, and since we only get a single local maximum, it must be an absolute maximum. To find the corresponding y-value, we go back to the equation that $4xy + 3x^2 = 36$ and plug 2 in for x, giving us that $8y + 12 = 36$, which we solve to get $y = 3$. Finally, to find the maximum volume, we just substitute our x and y values into $V = x^2 y$ to get that $V = (2)(2)(3) = 12$ cubic feet.

16. Let $f(x) = x^3 + 4x - 7$.

    (a) Use the Intermediate Value Theorem to show that $f(x)$ has at least one root.

    First of all, f is a polynomial, and so it is continuous. **It is really important to say this, because otherwise the Intermediate Value Theorem does not apply!** We note that $f(0) = -7$ and $f(2) = 9$, and so f has both negative and positive outputs. Then, since $f(0) < 0 < f(2)$, the Intermediate Value Theorem says that for some c such that $0 < c < 2$, we have $f(c) = 0$. In other words, $f(x)$ has a root on the interval $(0, 2)$.

(b) Use the Mean Value Theorem or Rolle's Theorem to show that $f(x)$ has at most one root.
Since f is a polynomial, it is continuous and differentiable everywhere, and so Rolle's Theorem applies. Suppose that $f(x)$ has two or more roots, and pick two of them, say $x = a$ and $x = b$. In other words, $f(a) = f(b) = 0$. Then by Rolle's Theorem, $f'(c) = 0$ for some value of c. On the other hand, we note that $f'(x) = 3x^2 + 4$, and therefore, $f'(x) \geq 4$ for all x. Then it cannot happen that $f'(c) = 0$, and having arrived at a contradiction, that means that our assumption was wrong. So $f(x)$ does not have two or more roots; in other words, it has at most 1 root.

17. Let $g(x) = e^x + x + 2$.

(a) Use the Intermediate Value Theorem to show that $g(x)$ has at least one root.
The function $g(x)$ is the sum of e^x and $x + 2$, both of which we know to be continuous, and thus g is itself continuous. So we may apply the Intermediate Value Theorem. We note that $g(-10) = e^{-10} - 10 + 2 = e^{-10} - 8$, and since $e^{-10} = 1/e^1 0$, that part is a very small number, and so $e^{-10} - 8$ is negative. We also note that $g(0) = e^0 + 0 + 2 = 1 + 0 + 2 = 3$. Therefore, $g(-10) < 0 < g(0)$. Thus, the Intermediate Value Theorem says that for some c such that $-10 < c < 0$, we have $g(c) = 0$. In other words, $g(x)$ has a root on the interval $(-10, 0)$.

(b) Use the Mean Value Theorem or Rolle's Theorem to show that $g(x)$ has at most one root.
Again, g is the sum of e^x and $x+2$, both of which we know to be continuous and differentiable, and so Rolle's Theorem applies. Suppose that $g(x)$ has two or more roots, and pick two of them, say $x = a$ and $x = b$. In other words, $g(a) = g(b) = 0$. Then by Rolle's Theorem, $g'(c) = 0$ for some value of c. On the other hand, we note that $g'(x) = e^x + 1$, and since e^x is always positive, $g'(x) \geq 1$ for all x. Then it cannot happen that $g'(c) = 0$, and having arrived at a contradiction, that means that our assumption was wrong. So $g(x)$ does not have two or more roots; in other words, it has at most 1 root.

18. A spherical soap bubble is slowly shrinking. If its surface area is decreasing at a rate of 50 square millimeters per second, how quickly is the radius decreasing when the surface area is 1000 square millimeters?

The surface area of a sphere is related to the radius by the equation $S = 4\pi r^2$. We are told that the surface area is decreasing by 50 square millimeters per second, which means that $dS/dt = -50$. Our goal is to find dr/dt, and the way we find how dS/dt and dr/dt are related is to take the derivative of both sides of $S = 4\pi r^2$ with respect to t. On the left, we get dS/dt, and on the right we get $8\pi r \frac{dr}{dt}$. Therefore, $dr/dt = \frac{dS}{dt} \cdot \frac{1}{8\pi r}$. To finish this, we need to know what value of r to use. We want dr/dt when $S = 1000$, and using the equation $S = 4\pi r^2$ we get that $r = \sqrt{1000/(4\pi)} = \sqrt{250/\pi}$. Thus, we get:

$$\frac{dr}{dt} = -50 \cdot \frac{1}{8\pi\sqrt{250/\pi}} = \frac{-50}{40\sqrt{10\pi}} = \frac{-\sqrt{10}}{8\sqrt{\pi}}.$$

19. A car drives along an elliptical track. The track can be modeled by the equation $x^2 + 5y^2 = 14$, where x and y are measured in kilometers of distance from the center of the track. As the car passes the point $(3, 1)$, the x-coordinate is increasing at a rate of 1.5 km/min. How quickly is the y-coordinate changing at that point?

To find how dx/dt and dy/dt are related, we take the derivative of the given equation by t on both sides, yielding

$$2x\frac{dx}{dt} + 10y\frac{dy}{dt} = 0.$$

Since x is increasing by 1.5 km/min, we have $dx/dt = 1.5$, and it was given that $(x, y) = (3, 1)$, so substituting all that information yields

$$2(3)(1.5) + (10)(1)\frac{dy}{dt} = 0,$$

and thus $dy/dt = -9/10$.

20. Find the area of the region bounded by the curves $y = 2x^2$ and $y = 4 + x^2$.

    First, let us find the points of intersection. Setting the two equations equal to each other and solving gives us $2x^2 = 4 + x^2$, and so $x^2 = 4$, from which it follows that $x = \pm 2$. Those will be our bounds of integration. Now, to find the area between two curves, we need to integrate (top minus bottom). In this range, $y = 4 + x^2$ is the larger function; for example, when $x = 0$ it yields 4 whereas $2x^2$ only gives 0. So we get

$$\int_{-2}^{2} \left((4 + x^2) - 2x^2\right)\, dx = \int_{-2}^{2} (4 - x^2)\, dx.$$

    To solve that, we start by finding an antiderivative of the inside, and we get $4x - \frac{x^3}{3}$. Then we plug in 2 and -2 and subtract the two answers. This gives us

$$(4(2) - \frac{2^3}{3}) - (4(-2) - \frac{(-2)^3}{3}) = (8 - \frac{8}{3}) - (-8 + \frac{8}{3}) = \frac{32}{3}.$$

21. Find the area of the region bounded by the curves $y = e^x$, $y = x^2$, $x = 0$, and $x = 2$.

    In the given region, e^x is always bigger than x^2, and so when we do (top minus bottom) we will get $e^x - x^2$. The range of x-values is just 0 to 2, the lines that were given. So we get

$$\int_0^2 (e^x - x^2)\, dx = e^x - \frac{x^3}{3} \,|_0^2 = (e^2 - \frac{2^3}{3}) - (e^0 - \frac{0^3}{3}) = e^2 - \frac{11}{3}.$$

22. Consider the region bounded by the curves $y = 1 - x^2$ and $y = 0$. What is the volume of the solid obtained by rotating this region about the line $y = 0$?

    The curve $y = 1 - x^2$ is an upside-down parabola with its vertex at $y = 1$, and so the region is this sort of "hill" shape. When we rotate that around the line $y = 0$, the the vertical cross-sections are discs. To find the volume, we need to integrate πr^2, where r is the radius of these discs, as a function of x. For a fixed x-value, the disc we get has its center on the x-axis (the line $y = 0$, around which we are rotating the region), and the top of the disc will be on the curve $y = 1 - x^2$. Therefore, the radius of the disc is $r = 1 - x^2$. We also need to find the limits for our integral, which we get by finding the x-values that are furthest to the left and to the right of our region. We do that by solving for the points of intersection of $y = 1 - x^2$ and $y = 0$; setting those equations equal and solving yields $x = \pm 1$. Thus, the integral we need to solve is

$$\int_{-1}^{1} \pi(1 - x^2)^2\, dx.$$

    To integrate that, we first multiply out $(1 - x^2)^2$, yielding $1 - 2x^2 + x^4$. So, we get

$$\int_{-1}^{1} \pi(1 - 2x^2 + x^4)\, dx = \pi(x - \frac{2x^3}{3} + \frac{x^5}{5}) \,|_{-1}^{1} = \pi(1 - \frac{2}{3} + \frac{1}{5}) - \pi(-1 + \frac{2}{3} - \frac{1}{5}) = \pi(2 - \frac{4}{3} + \frac{2}{5}) = \frac{16\pi}{15}.$$

23. Consider the region bounded by the curves $y = x^2$ and $x = y^2$. What is the volume of the solid obtained by rotating this region about the line $y = 0$?

This time, when we rotate the region about the x-axis (the line $y = 0$), the vertical cross-sections are washers: discs with holes in the middle. Let us rewrite $x = y^2$ as $y = \sqrt{x}$; then our region has $y = x^2$ as its bottom boundary and $y = \sqrt{x}$ as its top boundary. To find the volume of the resulting solid, we need to integrate $\pi r_o^2 - \pi r_i^2$, where r_i is the inner radius (the radius of the hole), and r_o is the outer radius (the radius of the entire thing). Each washer has its center at the x-axis, and the highest point of the hole is given by x^2. The highest point of the entire washer is given by $\sqrt{x}$. So we have that $r_i = x^2$ and $r_o = \sqrt{x}$. Now, since the curves $y = x^2$ and $y = \sqrt{x}$ intersect at $x = 0$ and $x = 1$, those form our bounds of integration. Thus, we get

$$\int_0^1 (\pi(\sqrt{x})^2 - \pi(x^2)^2)\, dx = \int_0^1 (\pi x - \pi x^4)\, dx = \frac{\pi x^2}{2} - \frac{\pi x^5}{5} \,\Big|_0^1 = \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}.$$