Lagrangian immersions and the Floer homotopy type

Joint with T. Kragh

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Background

Arnol'd conjectured that every closed exact Lagrangian in a cotangent bundle is Hamiltonian isotopic to the zero section.

Recent results:

1. Exact Lagrangian embeddings have vanishing Maslov index (Kragh)
2. Exact Lagrangian embeddings are homotopy equivalences (Seidel, Fukaya-Seidel-Smith, Nadler, A.)

Two questions:

1. Can one improve the homotopy equivalence to a diffeomorphism?
2. Assuming the Lagrangians are diffeomorphic, is the embedding isotopic to the zero section?

Both questions have weaker versions. We will discuss the second question, and the notion of "regular Lagrangian homotopy."

Strategy

1. Assign to every Lagrangian immersion a stable bundle over the loop space.
2. The (*twisted*) Floer homotopy type of the immersion is the Thom spectrum of this bundle.
3. Construct a *Viterbo tranfer* map from the Floer homotopy type of the zero section to the twisted Floer homotopy of a Lagrangian *embedding*.
4. Use the compatibility of transfer and classical inclusion to show that the transfer map is a homotopy equivalence.

Classifying Lagrangian Immersions

From now on, Q is a closed smooth manifold. L is a manifold diffeomorphic to Q. The following is an application of Gromov's h-principle for Lagrangian immersions.

Lemma

$[L, U]$ classifies Lagrangian immersions of L in T^*Q, in the homotopy class of the 0 section, up to regular Lagrangian homotopy.

The easy direction (which is the only one we use) goes as follows: fix an embedding $Q \subset \mathbb{R}^N$, with normal bundle ν. Associate to an immersion the totally real subbundle

$$TL \oplus \nu$$

of the trivial $\mathbb{C}^N$ bundle over L. The space of totally real bundles is a torsor over $[L, U(N)]$. By the stability of the homotopy groups of unitary groups, this is indepedent of N.

A stable bundle over $\mathcal{L}L$

$$\begin{array}{ccccc} [L, U] & \longrightarrow & [\mathcal{L}L, \mathcal{L}U] & \xrightarrow{=} & [\mathcal{L}L, \Omega U \times U] \\ & & & & \downarrow \\ & & [\mathcal{L}L, \mathbb{Z} \times BU] & \longleftarrow & [\mathcal{L}L, \Omega U] \end{array}$$

We obtain a stable vector bundle η over the free loop space of L. We also have the bundle TL, which is pulled back by evaluation at 0. We shall consider the Thom spectrum:

$$\mathcal{L}L^{-TL+\eta}.$$

Remark

If η is Lagrangian regularly homotopic to the zero section, and TL is stably trivial , then η is trivial, and $\mathcal{L}L^{-TL+\eta}$ is $\mathcal{L}L$.

Proposition

$\mathcal{L}L^{\eta}$ is homotopy equivalent to the suspension spectrum of $\mathcal{L}L$ if and only η is trivial as a spherical bundle.

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Spectrum via generating functions

Kragh produces a sequence of finite dimensional approximations $\Lambda^r T^*Q$ of $\mathcal{L}T^*Q$. Given a Hamiltonian function on T^*Q which is linear at infinity, he defines functions

$$\mathcal{A}_r \colon \Lambda^r T^*Q \to \mathbb{R}$$

which approximate the action functional. Let ν be the normal bundle of Q. The Floer homotopy type is defined to be

$$\lim_r (\Lambda^r T^*Q / \mathcal{A}_r^{-1}(-\infty))^{\nu - TQ}.$$

Proposition (Kragh)

The Floer homotopy type is homotopy equivalent to

$$\mathcal{L}Q^{-TQ}$$

The transfer map

The homotopy type depends only on the function at infinity. In particular, given an exact Lagrangian embedding

$$L \subset T^*Q$$

we may choose the Hamiltonian in a neighbourhood of L to be linear with respect to a Weinstein neighbourhood.

By choosing the function appropriately, Kragh ensures that the critical points of $\mathcal{A}_r$ with positive value are exactly the critical points of $\mathcal{A}_r|\Lambda^r D^*L$. If the Lagrangian foliation of T^*Q restricts to a foliation homotopic to the Lagrangian foliation of T^*L, we obtain a map

$$\Lambda^r T^*Q/\mathcal{A}_r^{-1}(-\infty) \to \Lambda^r T^*Q/\mathcal{A}_r^{-1}(-\infty, 0) \cong \Lambda^r T^*L/\mathcal{A}_r^{-1}(-\infty).$$

In general, the difference between these foliations introduces a twist. Taking the limit $r \to +\infty$, Kragh obtains the transfer map:

$$\mathcal{L}Q^{-TQ} \to \mathcal{L}L^{-TL+\eta}$$

The comodule structure

The diagonal map $X \to X \wedge X$ makes every space into a coalgebra. If E is a vector bundle on X, we have a map

$$X^E \to X \wedge X^E$$

which makes X^E into a comodule.
The projection map $L \to Q$ induces a map $\mathcal{L}L \to \mathcal{L}Q$.

Proposition (A-Kragh)

The transfer map is a map of comodules.

This is the space level refinement of work of Viterbo on *tied cohomology classes*. It is the key step in the proof of

Theorem (A-Kragh)

The transfer map is a homotopy equivalence.

A sketch of proof

$$
\begin{array}{ccccc}
\mathcal{L}Q^{-TQ} & & \xrightarrow{\quad \mathcal{L}j_! \quad} & & \mathcal{L}L^{-TL+\eta} \\
\downarrow & & & & \downarrow \\
\mathcal{L}Q_+ \wedge \mathcal{L}Q^{-TQ} & \xrightarrow{\mathrm{id} \wedge \mathcal{L}j_!} & \mathcal{L}Q_+ \wedge \mathcal{L}L^{-TL+\eta} & \xleftarrow{\mathcal{L}j \wedge \mathrm{id}} & \mathcal{L}L_+ \wedge \mathcal{L}L^{-TL+\eta} \\
\downarrow{\scriptstyle \mathrm{id} \wedge p_Q} & & \downarrow{\scriptstyle \mathrm{id} \wedge p_L} & & \downarrow{\scriptstyle \mathrm{id} \wedge p_L} \\
\Sigma^{-n}\mathcal{L}Q_+ & \xrightarrow{\quad \mathrm{id} \quad} & \Sigma^{-n}\mathcal{L}Q_+ & \xleftarrow{\Sigma^{-n}\mathcal{L}j} & \Sigma^{-n}\mathcal{L}L_+
\end{array}
$$

A computation

We obtain the following result for spheres ($k = n - 1$)

$k \bmod 8$	0	1	2	3	4	5	6	7
$\pi_k(\mathbb{Z} \times BU)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$\pi_k(\mathbb{Z} \times BO)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0
$im((\Omega q)_*)$	$2\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0
$im((BJ)_*)$	$\mathbb{Z}/c_k$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/c_k$	0	0	0
$im((BJ \circ \Omega q)_*)$	$\mathbb{Z}/c'_k$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/c_k$	0	0	0