The Crepant Resolution Conjecture and The Topological Vertex

- $\mathcal{X}$ is an orbifold Calabi-Yau threefold.

  Smooth Deligne-Mumford stack over $\mathbb{C}$ with $\dim_{\mathbb{C}} \mathcal{X} = 3$ $\quad c_1(T\mathcal{X}) = 0$. Locally modelled on $\mathbb{C}^3/G$ $\quad G \subset SU(3)$ finite subgroup

- X the underlying space of $\mathcal{X}$ is singular.

A resolution $Y \longrightarrow X$ is **crepant** if Y is a (smooth) Calabi-Yau.

- There exists a canonical crepant resolution

  (not true in dim > 3)

- There are two(*) conjecturally equivalent ways to count holomorphic curves on $\mathcal{X}$. **Gromov-Witten** theory and **Donaldson-Thomas** theory.
- The **Crepant Resolution Conjecture** concerns the relationship between the GW or DT theories of $\mathcal{X}$ and the crepant resolution Y

$$
\begin{tikzcd}
DT(Y) \arrow[r, dash, "\text{DT/GW correspondence}"'] \arrow[d, dash, "\text{DT CRC}"'] & GW(Y) \arrow[d, dash, "\text{GW CRC}"] \\
DT(\mathcal{X}) \arrow[r, dash, "\text{orbi-DT/GW correspondence.}"] & GW(\mathcal{X})
\end{tikzcd}
$$

(*) Pandharipande & Thomas count 13/2 ways

Donaldson-Thomas invariants of $\mathcal{X}$:

$$\mathrm{Hilb}^\alpha(\mathcal{X}) = \{ C \subset \mathcal{X} \text{ subscheme (substack) with } [\mathcal{O}_C] = \alpha \in K(\mathcal{X}) \}$$

$K(\mathcal{X})$ K-theory of $\mathcal{X}$. $F_d K(\mathcal{X}) \subset K(\mathcal{X})$

Classes supported in d dimensions

$$DT_\alpha(X) = \#_{vir} \mathrm{Hilb}^\alpha(\mathcal{X}) = \mathrm{euler}(\mathrm{Hilb}^\alpha(\mathcal{X}), \nu)$$

$$= \sum_{k \in \mathbb{Z}}{}' \, k \, \mathrm{euler}(\nu^{-1}(k))$$

Behrend function

$\nu : \mathrm{Hilb}^\alpha(\mathcal{X}) \to \mathbb{Z}$

assigns weights to singularities.

$$DT(\mathcal{X}) = \sum_{\alpha \in F_1 K(\mathcal{X})}{}' DT_\alpha(\mathcal{X}) \, q^\alpha$$

$$DT_0(\mathcal{X}) = \sum_{\alpha \in F_0 K(\mathcal{X})}{}' DT_\alpha(\mathcal{X}) \, q^\alpha$$

Crepant Resolution Conjecture for DT

Let $Y \longrightarrow X$ be the canonical crepant resolution

Fourier-Mukai isomorphism: $\mathrm{FM}: K(\mathcal{X}) \longrightarrow K(Y)$

Assume local models are $\mathbb{C}^3/G$ $\quad G \subset {SU(2) \atop SO(3)} \subset SU(3)$

Then $\mathrm{FM}(F_0 K(\mathcal{X})) \subset F_1 K(Y)$. Define:

$$DT_{exc}(Y) = \sum_{\alpha \in \mathrm{FM}(F_0K(\mathcal{X}))}' DT_\alpha(Y)\, q^\alpha \qquad DT_{reg}(\mathcal{X}) = \sum_{\alpha \in \mathrm{FM}^{-1}(F_1K(Y))}' DT_\alpha(\mathcal{X})\, q^\alpha$$

DT CRC:

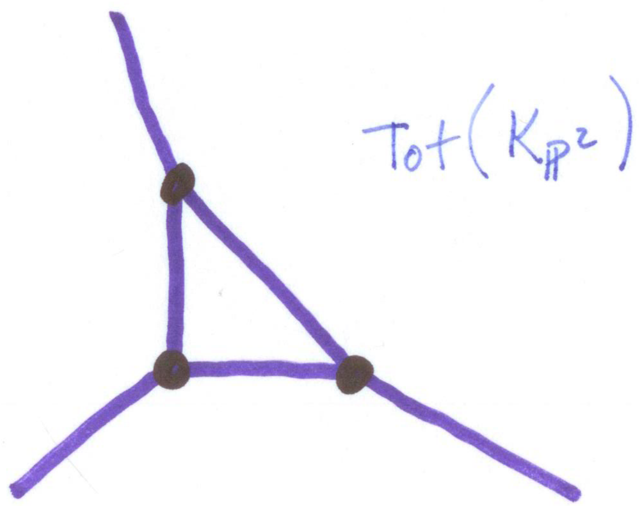
$$\frac{DT(Y)}{DT_{exc}(Y)} = \frac{DT_{reg}(\mathcal{X})}{DT_0(\mathcal{X})}$$

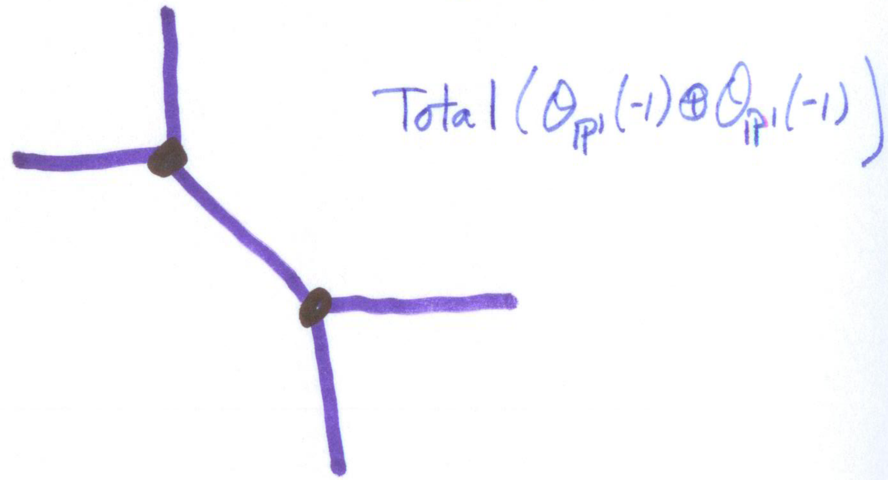
Conjectured by B-Cadman-Young 2008
Proved by Calabrese 2012.

$DT_0(\mathcal{X})$ & $DT_{exc}(Y)$ have closed universal formulas.

Computing $DT(\mathcal{X})$ for $\mathcal{X}$ toric.

The torus fixed points and torus invariant curves in $\mathcal{X}$ define a trivalent planar graph





Additional data determines how local charts $\mathbb{C}^3/G$ $(G \subset (\mathbb{C}^\times)^3)$ are glued (normal bundles of compact curves).

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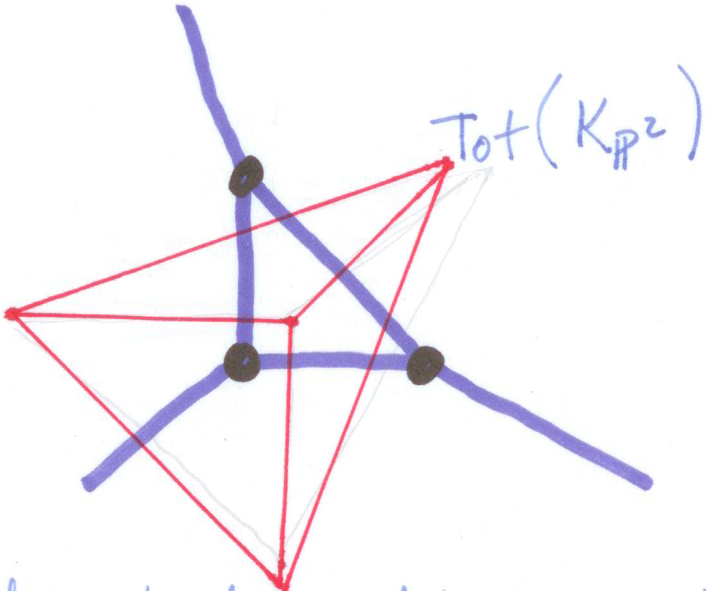

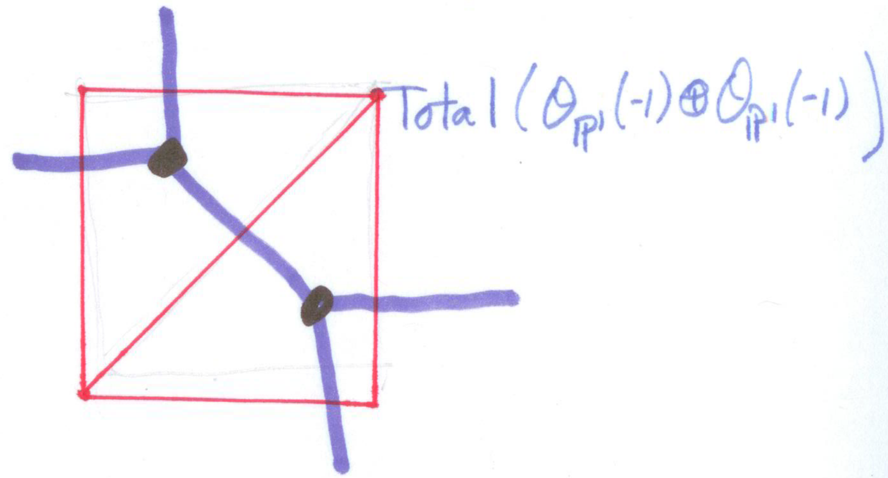

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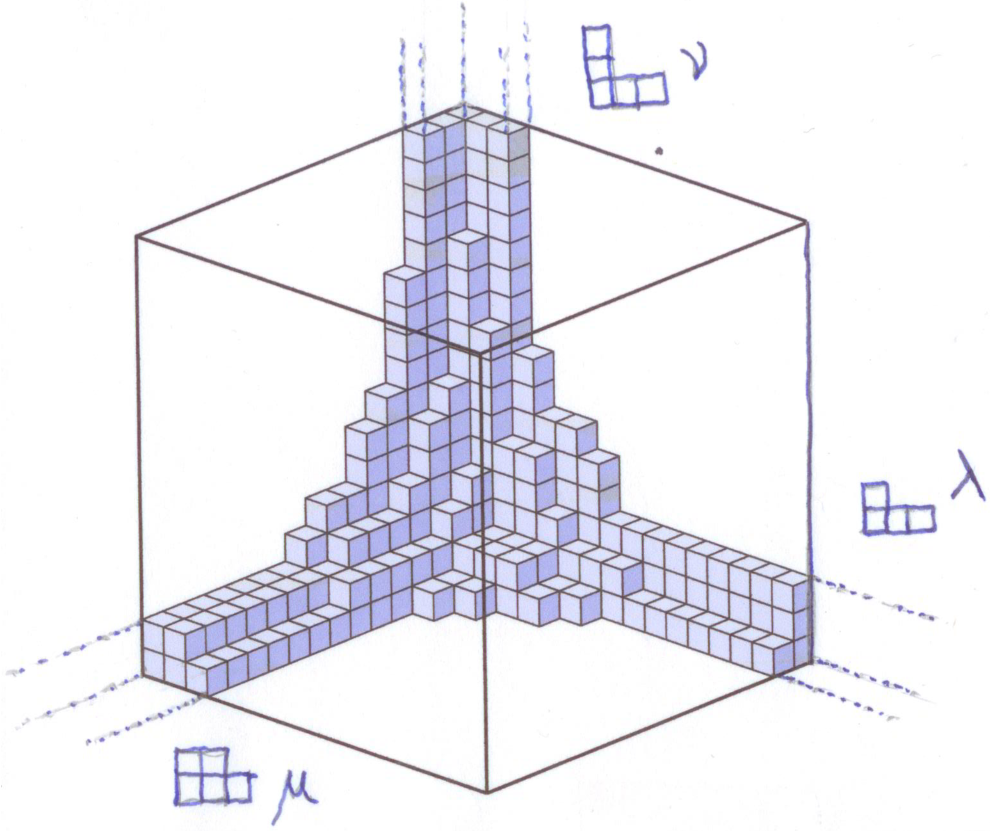
- Toric fan is the cone over the dual graph

- Induced torus action on $\mathrm{Hilb}^\alpha(\mathcal{X})$ has <u>isolated</u> fixed points
- $\nu(p) = \pm 1$ for torus fixed $p \in \mathrm{Hilb}^\alpha(\mathcal{X})$
- $DT_\alpha(\mathcal{X}) = \mathrm{euler}\left(\mathrm{Hilb}^\alpha(\mathcal{X}), \nu\right) =$ signed count of fixed points.

$$\left\{\begin{array}{l}\text{Torus fixed sub-}\\ \text{schemes of } \mathbb{C}^3/G\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{monomial ideals}\\ I \subset \mathbb{C}[x,y,z]\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{3D partitions } \pi\\ \text{(piles of boxes)}\end{array}\right\}$$

$$I \longleftrightarrow \pi = \left\{(i,j,k) \in \mathbb{Z}^3_{\geq 0} \;\; x^i y^j z^k \notin I\right\}$$

- Subschemes supported on curves $\Rightarrow$ boxes can extend to infinity along axes.



A 3D Partition with legs of type $(\lambda, \mu, \nu) = (\; , \; , \;)$

The **Topological Vertex** is a generating function for counting 3D partitions having legs of fixed type:

$$V_{\lambda\mu\nu}(q) = \sum_{\substack{\text{3D partitions } \pi \\ \text{with legs of type } (\lambda,\mu,\nu)}} q^{|\pi|}$$

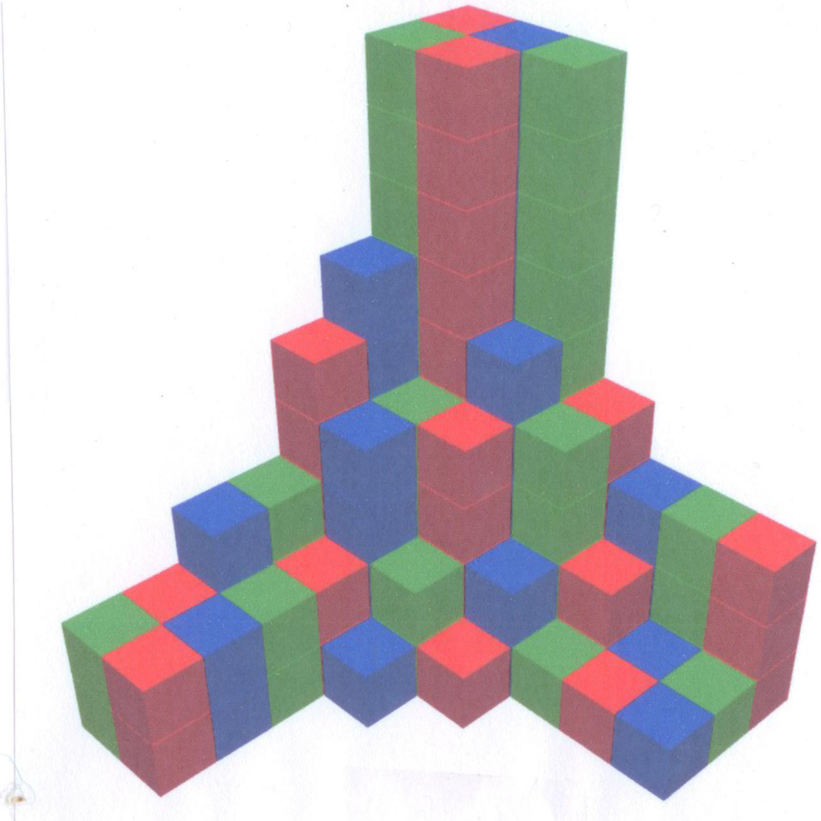
$$|\pi| = \text{normalized volume of } \pi = \sum_{(i,j,k)\in\pi} \big(1 - \#\text{ of legs containing } (i,j,k)\big)$$

For the orbifold $\mathbb{C}^3/G$

Boxes are "colored" by elements of $\hat{G}$ (characters of G).

<u>Example</u>: $G \cong \mathbb{Z}/3$ acting by $(x,y,z) \mapsto (e^{2\pi i/3}x, e^{-2\pi i/3}y, z)$

color of box (i,j,k) is $i-j \bmod 3$



The orbifold Vertex:

$$V^G_{\lambda\mu\nu}(q_0,\cdots,q_{n-1}) = \sum_{\substack{\text{3D partitions } \pi \\ \text{having legs of type } (\lambda\mu\nu)}}' q_0^{|\pi|_0} q_1^{|\pi|_1} \cdots q_{n-1}^{|\pi|_{n-1}}$$

$|\pi|_k$ = (normalized) number of boxes of color $k \in \hat{G}$.

An edge assignment is a choice of a 2D partition for each compact edge in the graph of $\mathcal{X}$.

Theorem (B.-Cadman-Young, non-orbifold MNOP)

$$DT(\mathcal{X}) = \sum_{\text{edge assignments}}' \prod_{e \in \text{Edges}} E_{\lambda(e)} \prod_{v \in \text{Vertices}} V^{G(v)}_{\lambda(v)\,\mu(v)\,\nu(v)}$$

simple factor, depends on geometry of $\mathcal{X}$

universal term given by vertex, complexity lies here

Theorem (BCY, non-orbifold: ORV)

$V^{\mathbb{Z}/n}_{\lambda\mu\nu}$ has an explicit formula in terms of loop schur functions

DT | GW Crepant Resolution Conjecture at the level of the topological vertex

Counting Boxes

Integrals of Hodge classes on the Moduli space of curves

$$\begin{tikzcd}
DT(Y) \arrow[r, dash] \arrow[d, dash] & GW(Y) \arrow[d, dash] \\
DT(\mathcal{X}) \arrow[r, dash] & GW(\mathcal{X})
\end{tikzcd}$$

Counting Boxes Colored by $\hat{G}$

Integrals of Hodge classes on the Moduli space of curves equipped with a G-action