Symplectic geometry in degree 2

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MCA 2013, Guanajuato

Outline:

1. Motivation: Generalized (complex) geometry...
2. N-manifolds
3. Symplectic N-manifolds
4. Hamiltonians
5. Building a dictionary
6. Applications

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Symmetries: $\mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$, $\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$

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- Cartan-Dirac on Lie groups...

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More: Generalized Kähler structures (= bihermitian geometry)

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Exact: $T^*M \to E \to TM$

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Moment maps: $\mu : M \to \mathfrak{h}^*$, $\mathfrak{h}$ a G-module...

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We will see:

Generalized geometry is *symplectic geometry on "deg 2 manifolds"*

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Differential calculus: vector fields (derivations), submanifolds (ideals) ...

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Example: deg. 1 vector field Q, $[Q, Q] = 2Q^2 = 0$

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Following Severa, Roytenberg, Vaintrob...

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Theorem: $(\mathcal{M},\omega)$ deg. 2 $\rightleftharpoons$ $(E, \langle\cdot,\cdot\rangle)$

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Symplectic N-manifold $(\mathcal{M}, \omega)$, degree r,

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What is lagrangian/coisotropic submanifold? Hamiltonian action? GCS? etc...

Continuing the dictionary: Coisotropic submanifolds

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where

- $C \hookrightarrow M$ submanifold
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- $F \subset TC$ integrable distribution
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Coisotropic reduction "$\mathcal{C}/\sim$" works when F is simple and ∇ has vanishing holonomy...

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Corollary: coisotropic reduction of Courant algebroids.

5. Dictionary (in degree 2)

$(E, \langle\cdot,\cdot\rangle)$	$(\mathcal{M}, \{\cdot,\cdot\})$ deg. 2, symplectic N-manifold
Courant structure $[\![\cdot,\cdot]\!]$, ρ	$\Theta \in \mathcal{A}_3(\mathcal{M})$, $\{\Theta,\Theta\} = 0$
$L \subset E\|_C$, $L = L^\perp$	$\mathcal{L} \subset \mathcal{M}$ Lagrangian submanifold
Dirac structure L (supp. on C)	$\mathcal{L}$ Lagrangian submanifold, $\Theta\|_{\mathcal{L}} \equiv$ const.
(C, K, F, ∇) such that ...	$\mathcal{C}$ coisotropic submanifold, Θ reducible
gen. complex struc. $\mathcal{J}$	$\mathcal{J} \in \mathcal{A}_2(\mathcal{M})$, $\{\{\Theta,\mathcal{J}\},\mathcal{J}\} = -\Theta$

Actions (DGLAs), moment maps, MW reduction ...

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- Homological methods (singular reduction)

Thank you

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- $\diamond$ Lagrangian submanifolds in $\mathcal{M}$ $\overset{1-1}{\rightleftharpoons}$ (C, L),
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Dirac structures (with support), Courant morphisms...