How to perturb the moduli space of constant maps

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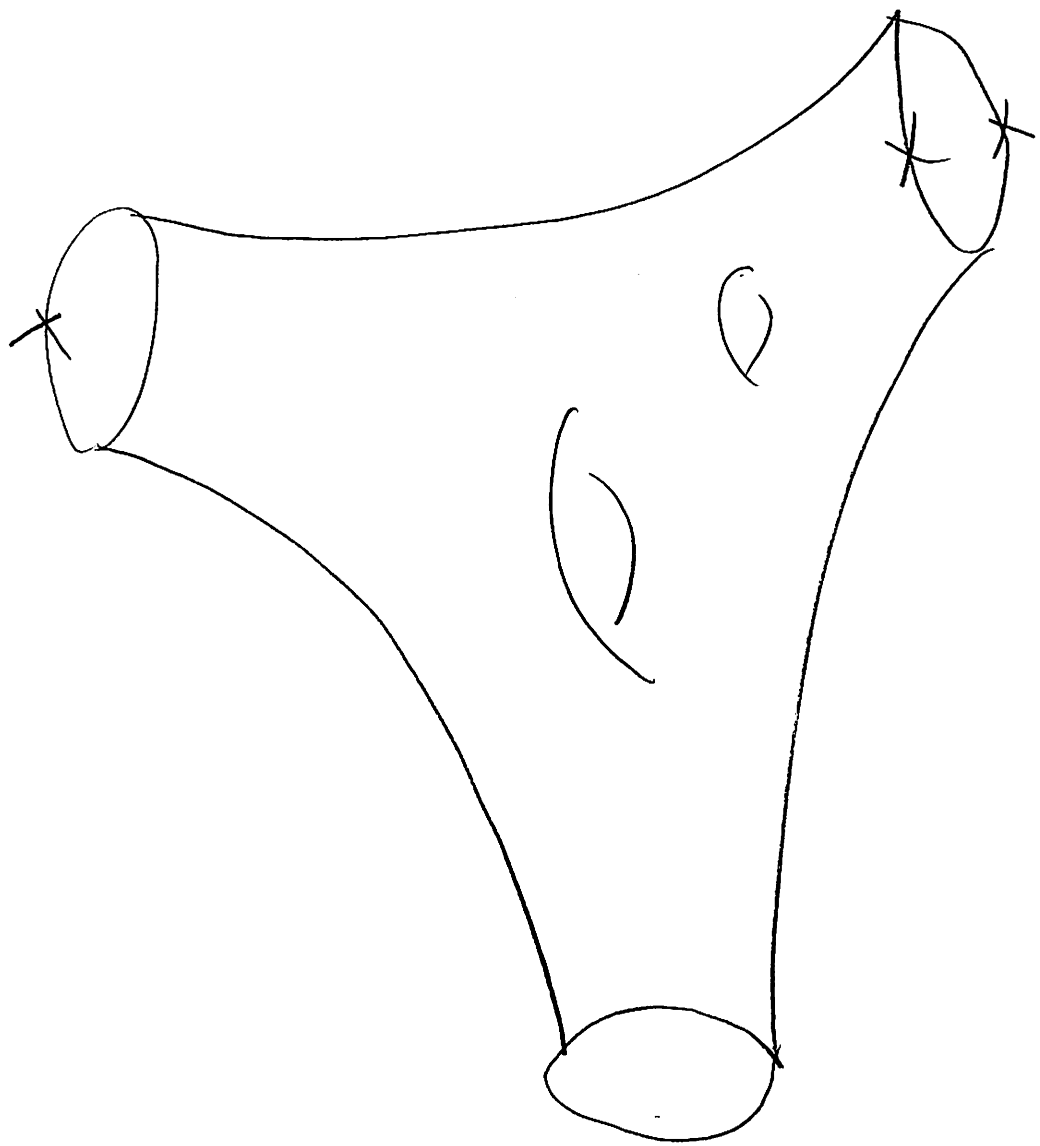
$\mathcal{M}_{g;(k_1,\cdots,k_\ell)}$ moduli space of $(\Sigma, \vec{\vec{z}})$ ①

Σ genus g $\quad \partial\Sigma = \ell$ union S^1

$\vec{\vec{z}} = (\vec{z}_1, \cdots, \vec{z}_\ell)$

$\vec{z}_i$: k_i pts on $\partial_i\Sigma$

$$\partial\Sigma = \coprod_{i=1}^{\ell} \partial_i\Sigma \qquad \partial_i\Sigma = S^1$$



$\in \mathcal{M}_{2,(2,1,0)}$

What new today.

The case $k_r = 0$ is included

if $\chi(L) = 0$.

③

$$\mathcal{M}_{g;(k_1 \cdots k_\ell)}(L;0) = \left\{ (\Sigma, \vec{z}, u) \;\middle|\; \begin{array}{l} (\Sigma, \vec{z}) \in \mathcal{M}_{g;(k_1 \cdots k_\ell)} \\ u : \Sigma \to L \end{array} \right\}$$

u: constant map

Actually we regard

$$u : (\Sigma, \partial \Sigma) \to (T^*L, L)$$

(+)

$\mathcal{M}_{g;(k_1,\dots,k_\ell)}(L;0)$ is $\mathcal{M}_{g;(k_1\cdots k_\ell)} \times L$

But this moduli space is highly non transversal

unless $g=0$, $\ell=1$.

$$\mathcal{M}_{0;k}(L;0) = D^{k-3} \times L$$

and is transversal.

⑤ <u>Thm</u> $\exists$ way to perturb $\mathcal{M}_{g;(k_1 \cdots k_\ell)}(L;U)$

in a consistent way to obtain

$$[\mathcal{M}_{g;(k_1 \cdots k_\ell)}(L;U)] = m_{g;(k_1 \cdots k_\ell)} \in \Omega(L^{\Sigma k_i})$$

differential form on $L^{\Sigma k_i}$

so that $m_{g;(k_1 \cdots k_\ell)}$ satisfies

Master equation.

(We assume $\chi(L)=0$ to include the case $k_i=0$.)

Frechet differential Involutive bi-Lie algebra ⑧

ΩL de Rham complex of L

$$B^k_{cyc}\Omega L = (\Omega L)^{\otimes k} / \mathbb{Z}_k$$

$$\pm x_1 \otimes \cdots \otimes x_k \mapsto x_k \otimes x_1 \otimes \cdots \otimes x_{k-1}$$

$\hat{B}^k_{cyc}\Omega L$ is a completion of $\mathrm{Hom}(B^k_{cyc}\Omega L)$.

$$
\begin{tikzcd}
\Omega(L^k) \arrow[rr, hook] & & \mathrm{Hom}\left((\Omega L)^{\otimes k}, \mathbb{R}\right) \\
\varphi(x_1 \cdots x_k) \arrow[rr, mapsto] & & \varphi_1 \otimes \cdots \otimes \varphi_k \arrow[r, mapsto] & \displaystyle\int_{L^k} \varphi(x_1 \cdots x_k)\varphi(x_1) \cdots \varphi(x_k) \\
\hat{B}^k_{cyc}\Omega L \arrow[uu, hook] & & & \\
\Omega(L^k) \cap \mathrm{Hom}\left(B^k_{cyc}\Omega L, \mathbb{R}\right) \arrow[u, "\text{Frechet closure}"'] \arrow[rr] & & \mathrm{Hom}\left(B^k_{cyc}\Omega L, \mathbb{R}\right) \arrow[uuu, hook]
\end{tikzcd}
$$

$$\hat{B}_{cyc}\Omega = \bigoplus_{k\geq 0} \hat{B}^{k}_{cyc}\Omega \Rightarrow \mathcal{C}$$

$$\hat{B}_{cyc}\Omega = \bigoplus_{k>0} \hat{B}^{k}_{cyc}\Omega \Rightarrow \mathcal{C}^{+}$$

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d exterior derivative induces

d on $\mathcal{C}$, $\mathcal{C}_{+}$

$\mathfrak{g}$ is (Fréchet) Bi-Lie

$$\{\ \} : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$\square : \mathfrak{g} \longrightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g}$$

completion

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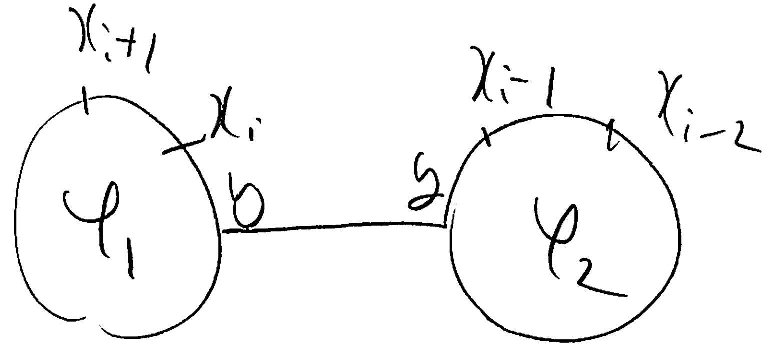
$\{\varphi_1, \varphi_2\} = \psi$ $\qquad \varphi_i \in \Omega(L^{k_i})$

$(\varphi_i \in \hat{B}_{cyc}(L))$

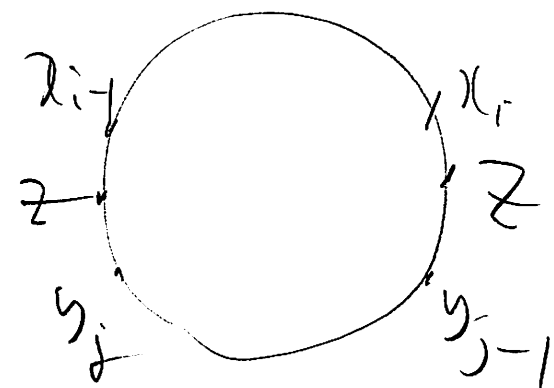
$\psi(x_1 \cdots x_{k_1+k_2-2})$

$$= \sum \pm \int_{y \in L} \varphi_1(y\, x_i \cdots)\, \varphi_2(y \cdots x_{i-1})$$



$$\varphi \in \hat{B}_{cyc}(\Omega_L) \subset \bigoplus \Omega(L^k)$$

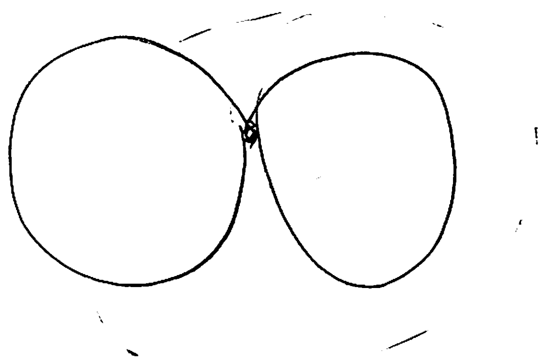
$$\Box\varphi \in \hat{B}_{cy}(\Omega_L) \,\hat{\otimes}\, \hat{B}_{cy}(\Omega_L) \subset \bigoplus \Omega(L^{k_1+k_2})$$



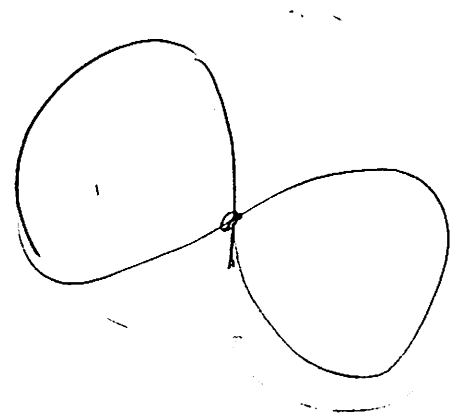
$$(\Box\varphi)(x_1 \cdots x_{k_1}, y_1 \cdots y_{k_2})$$

$$= \sum \pm \int_{z \in L} \varphi(z x_i \cdots x_{i-1} z y_j \cdots y_{j-1})$$

$\{\,,\}$, $\square$ are related to two operations of string topology



Bracket



cobracket

$(\mathcal{C}, \{\ \}, \square, d)$ is differential

Involutive Lie algebra (Frechet)

But we did not use wedge product

$\wedge$ of differential form

$$\wedge : \Omega L \otimes \Omega L \longrightarrow \Omega L$$

$$\rightsquigarrow$$

$$m_{0,(3)} \in \mathrm{Hom}(\Omega L^{\otimes 3}, \mathbb{R})$$

$$m_{0,(3)}(h_1, h_2, h_3) = \int_L h_1 \wedge h_2 \wedge h_3$$

$$(\Omega L, d, \wedge) \text{ is dGA } \text{“}\Longleftrightarrow\text{”} \begin{cases} d\, m_{0,(3)} = 0 \\ \{m_{0,(3)}, m_{0,(3)}\} = 0 \end{cases}$$

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However actually

$$m_{U(3)} \notin \hat{B}^{3}_{CY_3}(\Omega L)$$

Schwartz kernel is <u>not</u> C^∞ but is a distribution.

Cyclic A_∞ alg story

$$\exists\ m^{\varepsilon}_{0,(k)} \in \hat{B}^{k}_{cyc}(\Omega L) \subset \Omega L^{k}$$

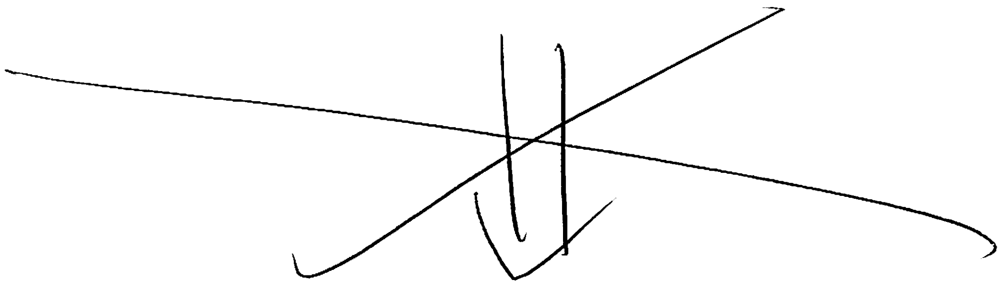
st

$$d\, m^{\varepsilon}_{0,(k)} + \sum_{k_1+k_2=k+2} \{ m^{\varepsilon}_{0,k_1}, m^{\varepsilon}_{0,k_2} \} = 0$$

This equation is equivalent to an A_∞ rel

However cyclic A_∞ str. on ΩL.

with C^∞ Schwartz kernel



Solution of Master equations

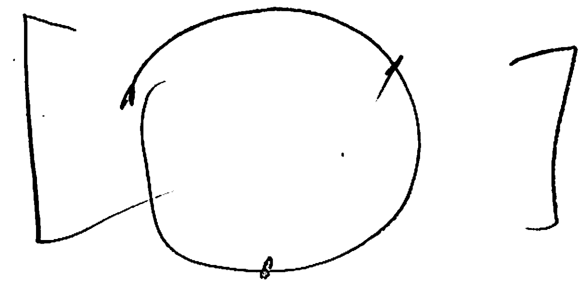
First 1-loop Master equation

$$d\,\mathfrak{m}_{1;(1,0)} + \square\,\mathfrak{m}_{0,(3)} = 0$$

wbracket

In case $\square\,\mathfrak{m}_{0,(3)}$ we need $\mathfrak{m}_{1;(1,0)}$.

$\mathcal{M}_{0;(3)} =$



virtual fundamental class

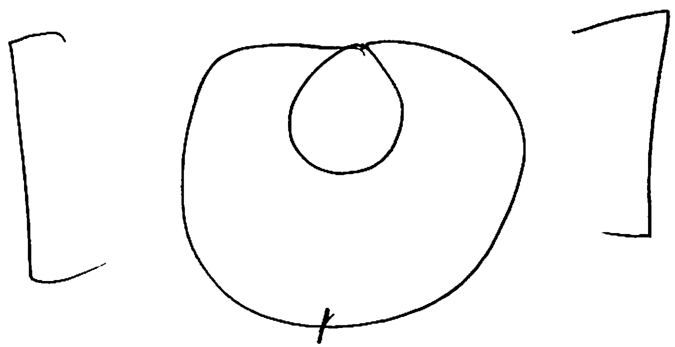
of $\mathcal{M}_{0,3}(L;0)$

$\|$

L

$$\mathcal{M}_{0,3}(L;0) \xrightarrow{ev} L^3$$

$\mathrm{ev}_! \left(\mathcal{M}_{0,3}(L;U) \right) =$ current realized by $L \subset L \times L \times L$.

$\square\, \mathfrak{m}_{U!(3)} =$ 

$$\mathcal{M}_{0,3}(L;0) = L \xrightarrow{\ i\ } L^3$$

$$x \longmapsto (x,x,x)$$

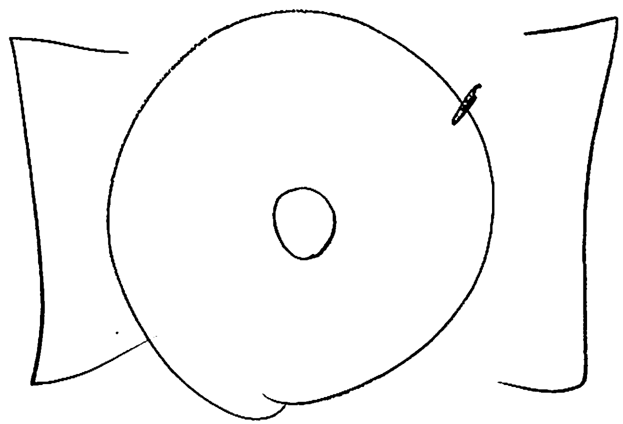
$$\begin{array}{c} L^3 \\ \cup\ j \\ L^2 = \{(y,y,z)\} \end{array}$$

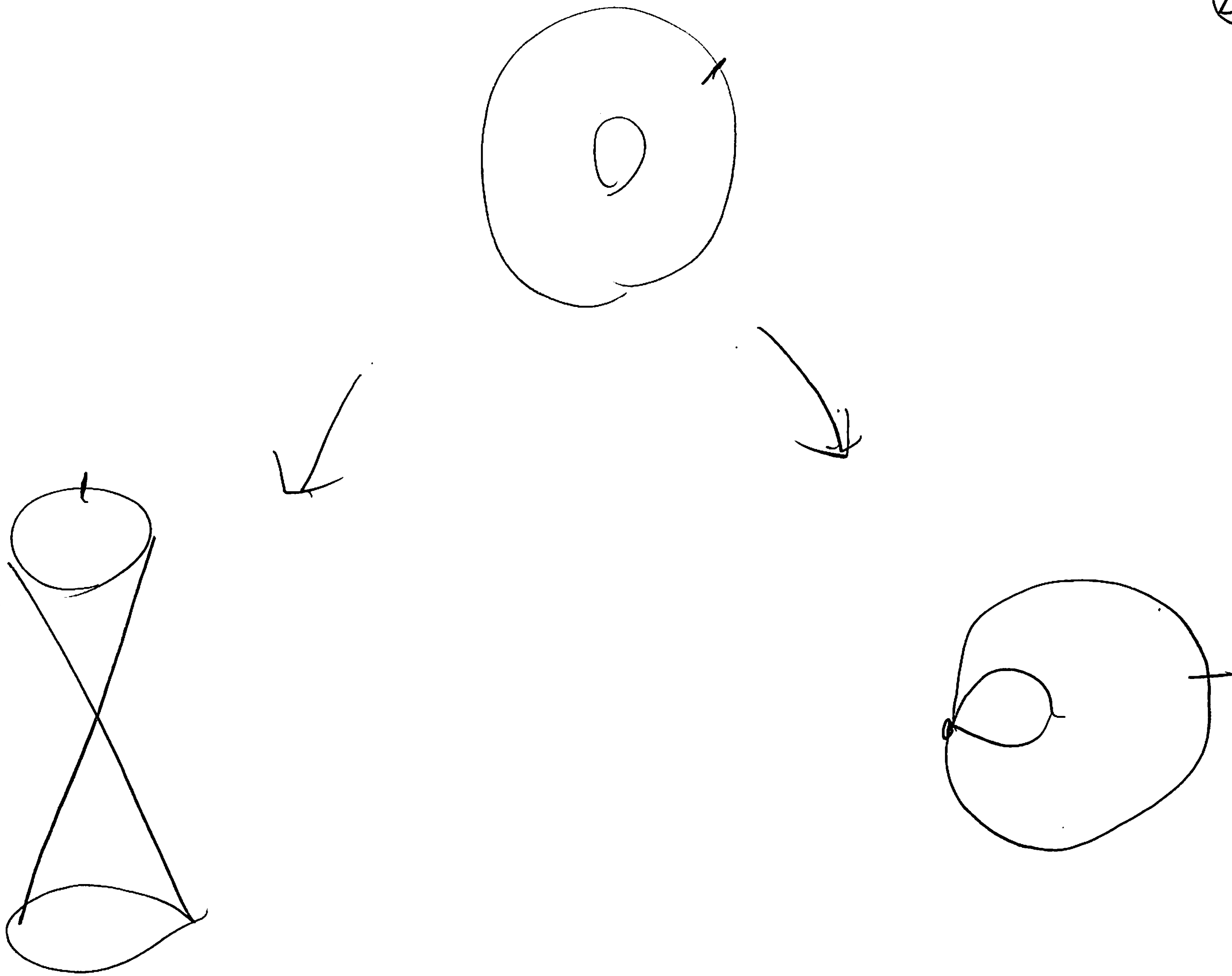
$$\square\mathcal{M}_{0;(3)}(L;0) = i(L) \cap j(L^2) \quad \text{in } L^3$$

This is highly nontransversal intersection.

However

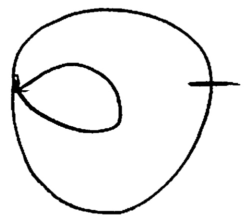
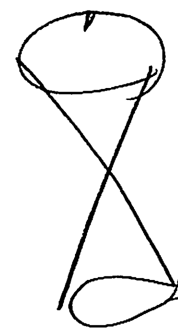
$$\square\mathfrak{M}_{0;(3)}(L;U) \subset \partial\mathcal{M}_{0;(1,v)}(L;U)$$

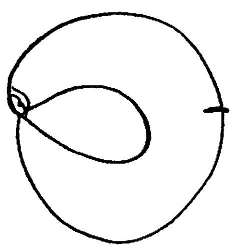
Note $\mathcal{M}_{0;(1,v)}(L;U) =$ 



Namely

$$M_{1;(1,0)} = [0,1]$$

$$\partial M_{1;(1,0)} = \Big\{$$  ,  $$\Big\}$$

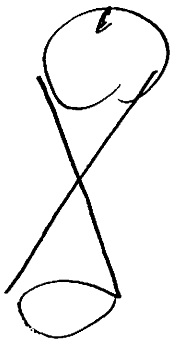
 corresponds □ $M_{0;3}$.

So let put

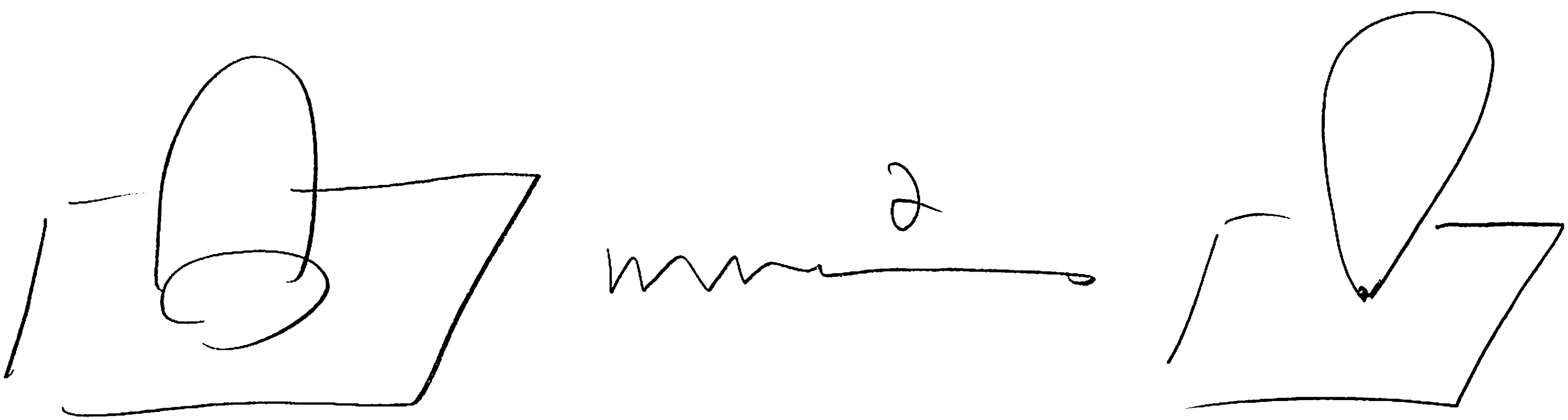
$$m'_{1,(1,0)} = [M_{1;(1,0)}],$$

$$d\, m'_{1,(1,0)} + \square\, m_{0,(3)} + \left[\right.$$  $$\left. \right] = 0$$

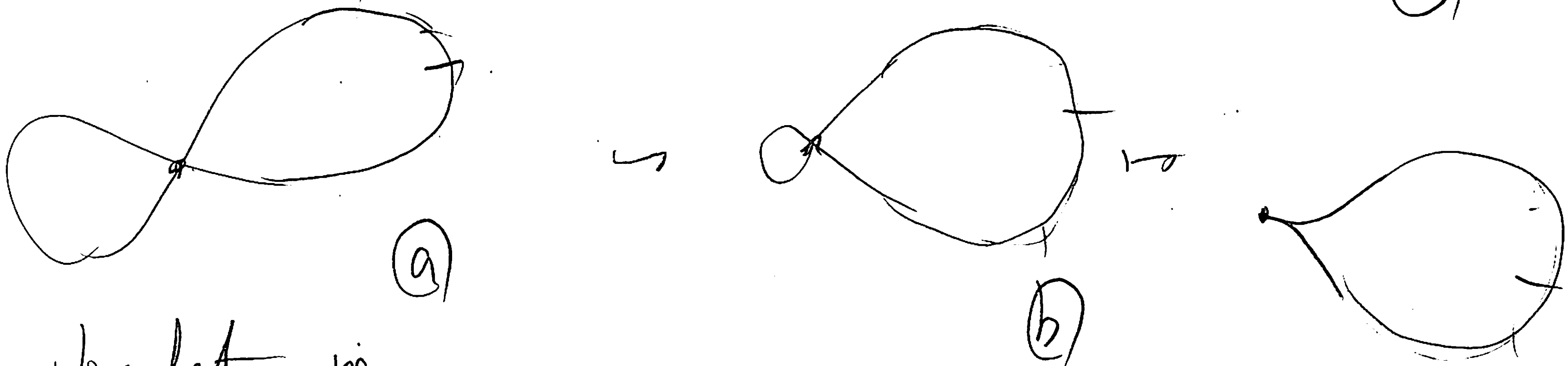
Master eq is __not__ yet satisfied



is a constant map version of a know phenomenon in disk counting

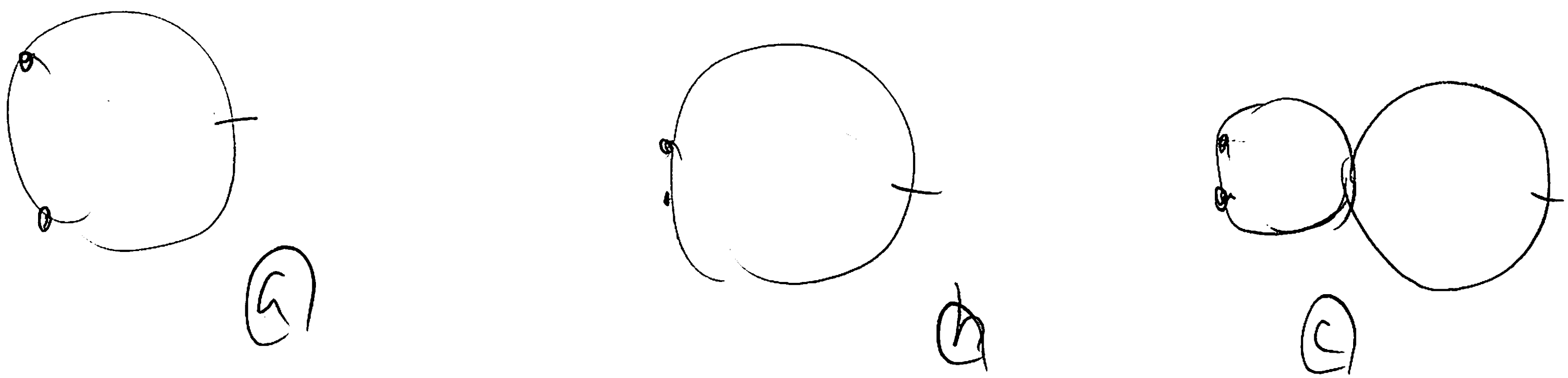


Note  is related to a known problem in String topology



cobracket in string topology

|| equivalent



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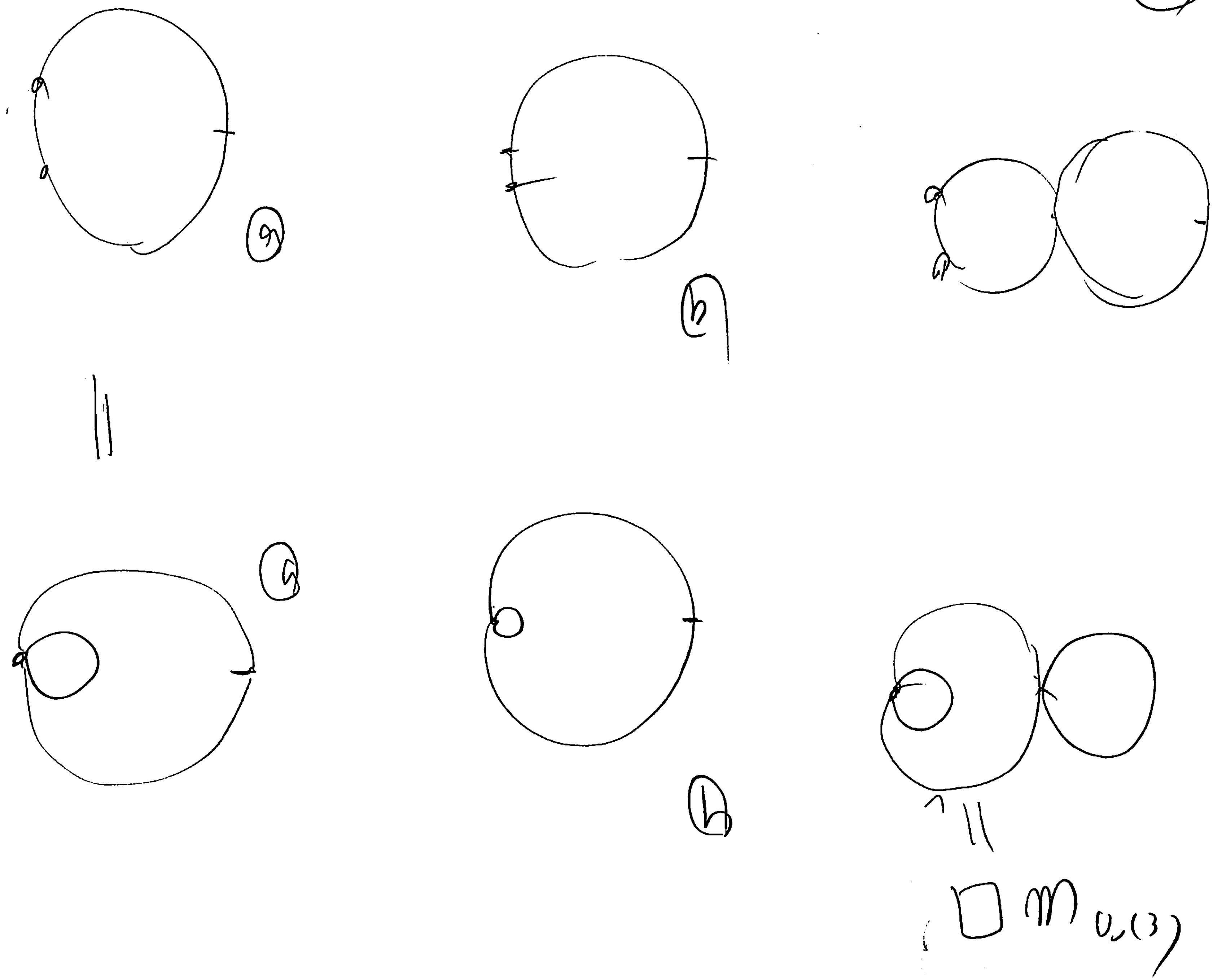
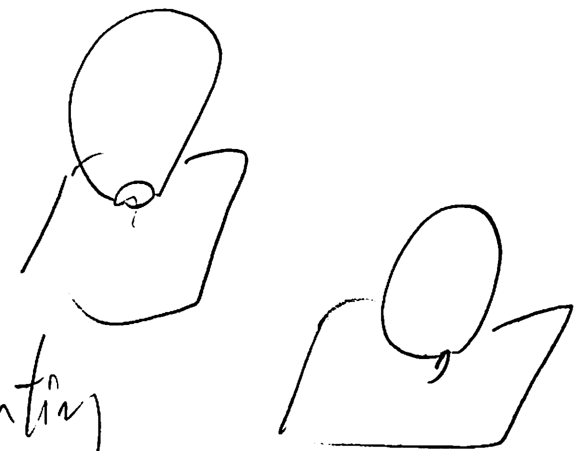


Observation

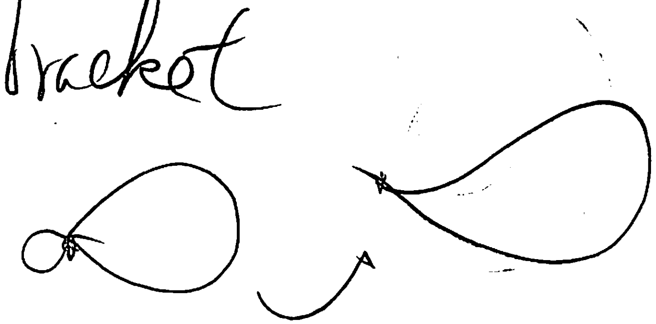


Main trouble of disk counting

is abundant to

main trouble to include constant map

in string topology bracket

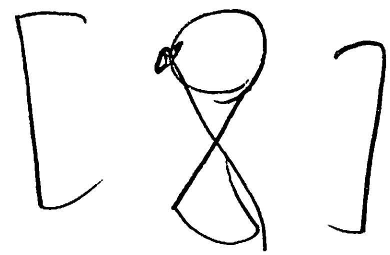


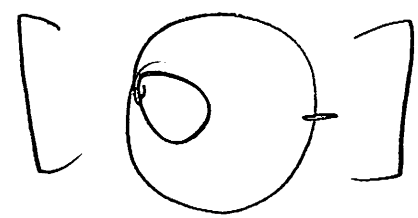
They both related to $\chi(L)$ euler number.

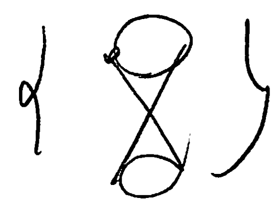
$$\begin{tikzcd} L \\ \mathcal{M}_{0;(1,0)}(L,0) \arrow[u, "ev"'] \arrow[d, "\pi"] \\ \mathcal{M}_{0;(1,0)} \end{tikzcd}$$

<u>Lemma</u> $w \in \mathcal{M}_{0;(1,0)}$

$ev_![\pi^{-1}(w)]$ is cohomologous to Euler form.

Note $ev_! [\pi^!(W)] =$ 

$=$  $\in H(L)$

Assuming $\chi(L) = 0$ we find a way to kill  from $\partial M_{0;C(L,U)}(L,U)$.

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$\chi(L) = 0$

$\Rightarrow$ section v of T^*L

nowhere vanish.

$K = \{ t v(x) \mid t \in \mathbb{R}_{\geq 0},\ x \in L \}$

$\subset T^*L$

$\partial K = L$

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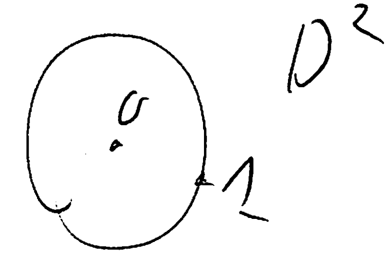
$$\left\{ (D^2, u, \gamma(u)) \;\middle|\; \begin{array}{l} u : (D^2, \partial D^2) \to (T^*L, L) \\ \text{holomorphic} \\ [u] = 0 \\ u(0) \in K \end{array} \right\} \Big/ \mathrm{Aut}$$

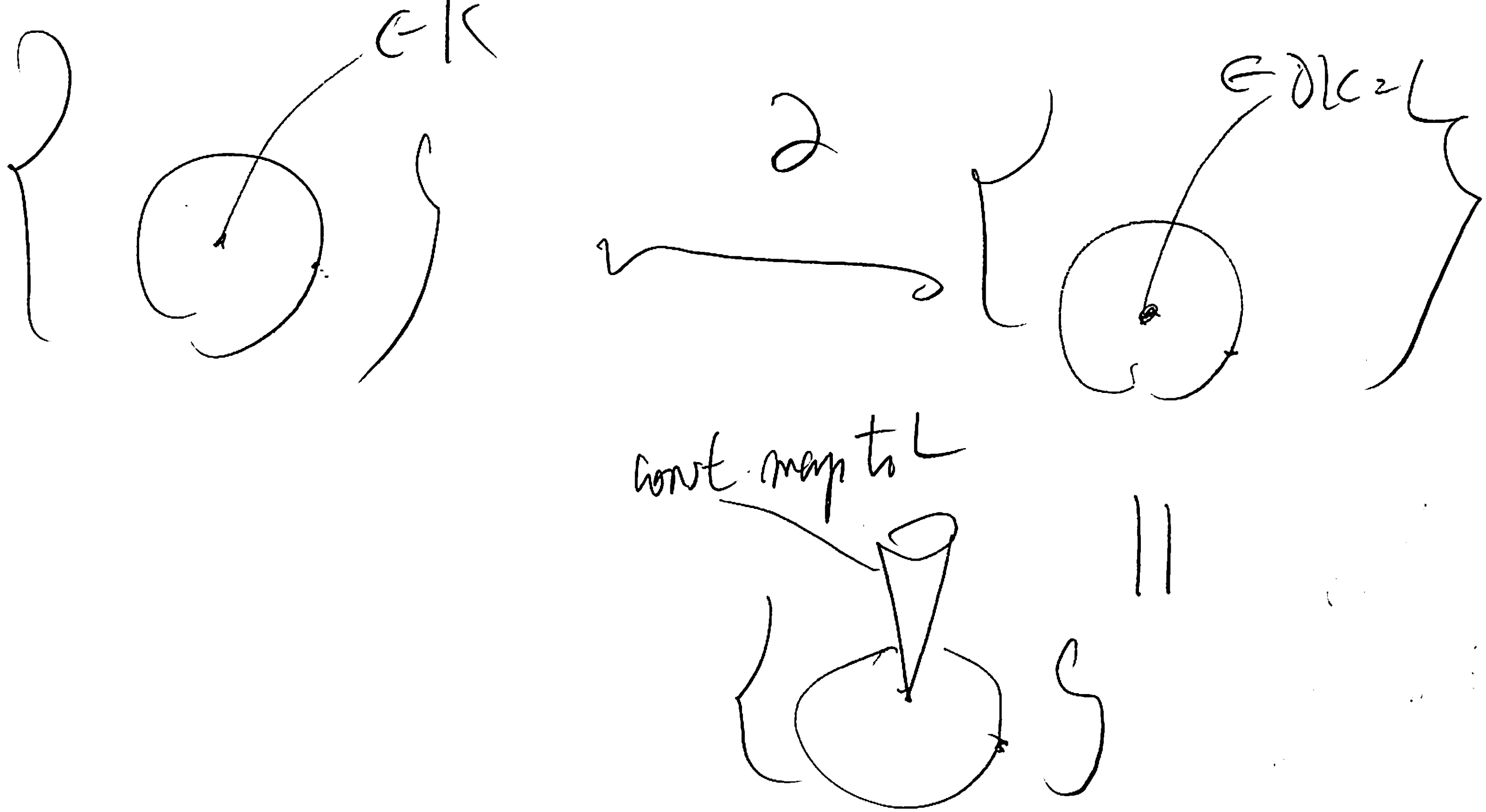


$\parallel$

A

$A = L$ but is highly nontransversal

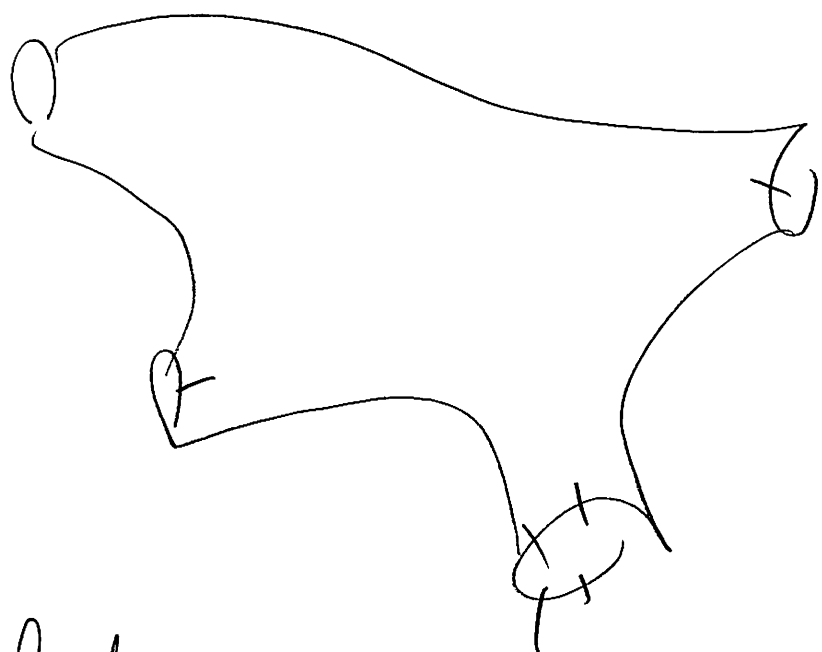
$\partial A \simeq \mathcal{M}_{1;(1,0)}(L;U)$ Red boundary of



(37)

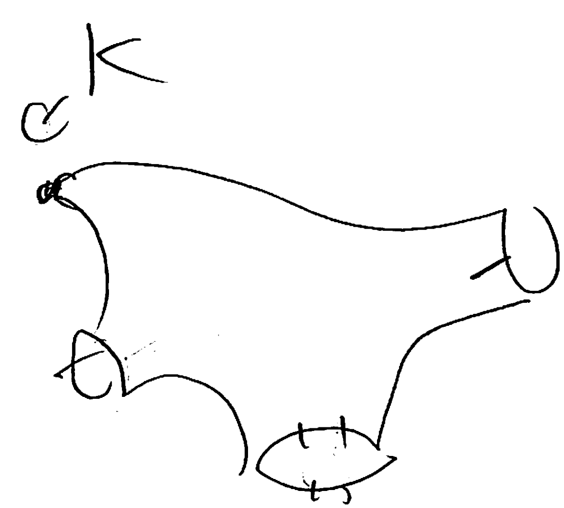
$\chi(L) = 0 \rightsquigarrow$ take $K \subset T^4 L$

$\partial K = L$



$\mathcal{M}_{S;(h_1 \cdots h_e)}$

$h_1 = 0$



include

Their virtual fundamental chains in de Rham theory satisfies Master equation in our Fréchet Involutive Li Lie alg.

<u>Note</u>

, k- dependence may be related to frame dependence of Perturbative Chern–Simons theory

Note

Can include nonconstant pseudo holomorphic curve in the same way

<u>Note</u>

We are developping de Rham theory version of virtual technique ~~~~ more thoroughly so that it can be applied safely in this very wild intersection theory.

Note

The construction looks very much related to Costello's work on renormalization theory

<u>Note</u>

My dream is to apply de Rham version of virtual technique to find a rigorous foundation of quantum field theory.