Extensions of the Weinstein conjecture in three dimensions

Michael Hutchings

Department of Mathematics
University of California, Berkeley

Mathematical Conference of the Americas
August 5, 2013

Outline

1. Review of the Weinstein conjecture
2. Improvements on the Weinstein conjecture
   - More Reeb orbits
   - Short Reeb orbits
   - More general vector fields
3. Embedded contact homology
4. Some proofs

Part 1

With apologies for boring the experts, we begin with a brief review of the Weinstein conjecture.

Closed orbits of vector fields on three-manifolds

Question

Let Y be a closed oriented three-manifold. Does every vector field on Y have a closed orbit?

It is easy to construct vector fields on T^3 with no closed orbit (irrational rotations), but hard on S^3.

Seifert conjecture

Every vector field on S^3 has a closed orbit.

- Counterxamples: P. Schweizer (C^1), J. Harrison (C^2), K. Kuperberg (C^∞), G. Kuperberg (C^1 and volume-preserving).
- The constructions involve inserting a “plug” to destroy a closed orbit without creating new ones.
- These methods yield vector fields with no closed orbit on any three-manifold.

Hamiltonian vector fields

Consider $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega = \sum_{i=1}^{n} dx_i dy_i.$$

A smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$ determines a Hamiltonian vector field X_H by the equation $dH = \omega(X_H, \cdot)$. More explicitly,

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} \right).$$

Note that X_H is tangent to each level set of H, because $\omega(X_H, X_H) = 0$.

Question

Let Y be a compact regular level set of $H : \mathbb{R}^{2n} \to \mathbb{R}$. Must X_H have a closed orbit on Y?

Answer

- Yes for almost every compact regular level set (Hofer-Zehnder, Struwe).
- But no in general. There are C^2 counterexamples $H : \mathbb{R}^4 \to \mathbb{R}$ by Ginzburg-Gürel (2003), and smooth counterexamples are known in higher dimensions.

Remark

If Y is a regular level set of H, then up to parametrization, closed orbits of X_H on Y depend only on Y and not on H. Indeed, embedded closed orbits are just leaves of the 1-dimensional characteristic foliation of Y defined by $\operatorname{Ker}(\omega|_Y)$. These are called closed characteristics of Y.

Reeb vector fields

Definition

Let Y be a closed smooth manifold of dimension $2n - 1$.
A contact form on Y is a 1-form λ on Y such that $\lambda \wedge (d\lambda)^{n-1} \neq 0$.

A contact form λ determines:

- A Reeb vector field R, defined by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$.
- A contact structure, namely the oriented $(2n - 2)$-plane field $\xi = \mathrm{Ker}(\lambda)$.

Definition

A Reeb orbit is a periodic orbit of R, i.e. a map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$, modulo reparametrization, such that $\gamma'(t) = R(\gamma(t))$.

Contact dynamics and Hamiltonian mechanics

Example

If Y is a star-shaped hypersurface in $\mathbb{R}^{2n}$, then

$$\lambda = \frac{1}{2}\sum_{k=1}^{n}(x_k dy_k - y_k dx_k)$$

restricts to a contact form on Y. The Reeb vector field is a rescaling of the Hamiltonian vector field for any Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ having Y as a regular level set.

Here a hypersurface Y in $\mathbb{R}^{2n}$ is called star-shaped if it is transverse to the radial vector field

$$\rho = \frac{1}{2}\sum_{k=1}^{n}\left(x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k}\right).$$

Contact dynamics and geodesic flow

Let M be a smooth manifold. The cotangent bundle T^*M has a canonical 1-form λ_{can} defined by

$$\lambda_{can}(q,p) = p \circ \pi_*$$

for $q \in M$ and $p \in T_q^*M$, where $\pi : T^*M \to M$ is the projection.

Example

Choose a Riemannian metric g on M. Then the restriction of λ_{can} to the unit cotangent bundle

$$ST^*M = \{(q,p) \in T^*M \mid |p| = 1\}$$

is a contact form. Under the identification $T^*M = TM$ determined by g, the Reeb vector field corresponds to the geodesic flow. Thus Reeb orbits are equivalent to (oriented) closed geodesics.

Weinstein conjecture (1978)

Every contact form on a closed oriented odd-dimensional manifold has a Reeb orbit.

Theorem (Taubes, 2006)

The Weinstein conjecture is true in three dimensions: every contact form on a closed oriented three-manifold has a Reeb orbit.

The proof uses Seiberg-Witten theory. In higher dimensions the Weinsten conjecture is open in general, although it is known in some cases.

Part 2

We now state some theorems and conjectures generalizing the Weinstein conjecture in three dimensions. In particular:

1. Can one improve the lower bound on the number of Reeb orbits?
2. Can one prove the existence of Reeb orbits with an *a priori* upper bound on the length?
3. Can we replace Reeb vector fields by any more general vector fields?

(Answers: 1. Yes. 2. Maybe. 3. Yes.)

More Reeb orbits

Theorem (with Taubes, 2008)

Let Y be a closed oriented three-manifold and let λ be a nondegenerate contact form on Y. Then:

- *λ has at least two embedded Reeb orbits.*
- *If Y is not S^3 or a lens space, then λ has at least three embedded Reeb orbits.*

Here a contact form is nondegenerate if all Reeb orbits are nondegenerate, or "cut out transversely", in the following sense. If $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ is a Reeb orbit, the Reeb flow for time T induces a "linearized return map", which is a symplectic linear map

$$P : (\xi_{\gamma(0)}, d\lambda) \to (\xi_{\gamma(0)}, d\lambda).$$

The Reeb orbit γ is nondegenerate if $1 \notin \mathrm{Spec}(P)$. This implies that it is isolated in the free loop space.

A generic contact form is nondegenerate.

Removing the nondegeneracy assumption

Theorem (with D. Cristofaro-Gardiner and V. Ramos, 2012)

Every (possibly degenerate) contact form on a closed oriented three-manifold has at least two embedded Reeb orbits.

This was new even for the standard contact structure on S^3; an alternate proof in that case was given by Ginzburg, Hein, Hryniewicz, and Macarini. In particular, this proves the $n = 2$ case of:

Old conjecture (Ekeland-Hofer?)

Every star-shaped hypersurface in $\mathbb{R}^{2n}$ has at least n closed characteristics.

Various special cases have been proved by Yiming Long et al.

The ellipsoid example

- Let $a, b > 0$ with a/b irrational. Identify $\mathbb{R}^4 = \mathbb{C}^2$. Define the ellipsoid

$$E(a,b) = \left\{ z \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \le 1 \right\}.$$

  Then $Y = \partial E(a,b)$ is star-shaped, so it has a canonical contact form.

- The Reeb vector field is

$$R = 2\pi \left(\frac{1}{a}\frac{\partial}{\partial\theta_1} + \frac{1}{b}\frac{\partial}{\partial\theta_2} \right).$$

  There are just two embedded Reeb orbits: the circles where $z_1 = 0$ or $z_2 = 0$.

- One can take the quotient by the action of a finite cyclic group to obtain a contact form on a lens space with only two embedded Reeb orbits.

Infinitely many Reeb orbits?

- Embarrassingly, the above are the only known examples of contact forms on closed three-manifolds with only finitely many embedded Reeb orbits.
- Hofer-Wysocki-Zehnder, 2003: A generic closed star-shaped hypersurface in $\mathbb{R}^4$ has infinitely many embedded Reeb orbits.
- Colin-Honda, 2008: For many contact structures ξ on closed oriented three-manifolds Y, every contact form λ with $\operatorname{Ker}(\lambda) = \xi$ has infinitely many embedded Reeb orbits. (Proof uses linearized contact homology.)

Conjecture

If Y is a closed oriented connected three-manifold which is not S^3 or a lens space, then every contact form on Y has infinitely many embedded Reeb orbits.

Short Reeb orbits

The length or symplectic action of a Reeb orbit γ is defined by

$$\mathcal{A}(\gamma) = \int_\gamma \lambda > 0.$$

Question

Given a closed contact three-manifold (Y, λ), does there exist a Reeb orbit with an *a priori* upper bound on the length?

Answer

Not without information, because one can multiply the contact form by a positive constant r, and this multiplies the lengths of all Reeb orbits by r.

But we can obtain a statement which is invariant under scaling as follows.

Define the volume of a contact 3-manifold (Y, λ) by

$$\mathrm{vol}(Y, \lambda) = \int_Y \lambda \wedge d\lambda.$$

Question

Given a closed contact 3-manifold (Y, ξ), does there exist $c(Y, \xi) \in \mathbb{R}$ such that every contact form λ with $\mathrm{Ker}(\lambda) = \xi$ has a Reeb orbit γ with

$$\mathcal{A}(\gamma)^2 \leq c(Y, \xi)\,\mathrm{vol}(Y, \lambda)$$

Short Reeb orbit conjecture

For any closed contact 3-manifold (Y, λ), there exists a Reeb orbit γ with

$$\mathcal{A}(\gamma)^2 \leq \mathrm{vol}(Y, \lambda).$$

We can (with Cristofaro-Gardiner and Ramos) prove a much weaker, "asymptotic" version of this which we will state later.

Example 1: a prequantization form times an S^1-invariant function

- Let Σ be a Riemann surface and let $\rho : Y \to \Sigma$ be a principal S^1-bundle with Euler class $e > 0$. Let ω be a symplectic form on Σ with area $2\pi e$. Let A be a connection 1-form on Y with curvature ω. Let $f : \Sigma \to \mathbb{R}$ be a positive function. Let $\lambda = (\rho^* f)A$. Then λ is a contact form on Y.
- If $x \in \Sigma$ is a critical point of f, then $\rho^{-1}(x) \subset Y$ is a Reeb orbit with action $\mathcal{A}(\rho^{-1}(x)) = 2\pi f(x)$. (There may also be other Reeb orbits.)
- The volume is given by

$$\operatorname{vol}(Y, \lambda) = 2\pi \int_\Sigma f^2 \omega \geq 4\pi^2 e \min(f).$$

- So the conjecture holds in this case, with equality if and only if $e = 1$ and f is constant.

Example 2: systolic inequalities

- Let g be a Riemannian metric on S^2 with area 4π. Let λ be the canonical contact form on STS^2 whose Reeb vector field is the geodesic flow. The symplectic action of a Reeb orbit agrees with the length of the corresponding geodesic.
- Let Y be the universal (double) cover of STS^2 and let $\widetilde{\lambda}$ be the pullback of the contact form. This has volume $16\pi^2$. Reeb orbits are "even" geodesics (geodesics which determine plane curves with odd rotation number). So the conjecture says that there is an even geodesic of length $\le 4\pi$.
- This is sharp for the round metric. (A great circle is an odd geodesic, but its double cover is even.)
- The Calabi-Croke example of two flat equilateral triangles glued together gives a metric on S^2 with area 4π and no closed geodesic of length $\le 2\pi$. However it still has an even geodesic of length $< 4\pi$. (Exercise.)

More general vector fields

Let Y be a closed oriented 3-manifold.

Definition

- A stable Hamiltonian structure on Y is a pair (λ, ω) where λ is a 1-form, ω is a closed 2-form, $\lambda \wedge \omega > 0$, and $d\lambda = f\omega$ where $f : Y \to \mathbb{R}$ is a smooth function.
- The associated "Reeb vector field" R is defined by $\omega(R, \cdot) = 0$ and $\lambda(R) = 1$.

Example

If λ is a contact form, then $(\lambda, d\lambda)$ is a stable Hamiltonian structure with $f \equiv 1$, and R agrees with the usual Reeb vector field.

Example

If ϕ is a symplectomorphism of a closed oriented symplectic 2-manifold (Σ, ω_Σ), then the mapping torus $\Sigma \to Y_\phi \to S^1$ has a stable Hamiltonian structure with ω determined by ω_Σ, and λ pulled back from the canonical 1-form on S^1. Here $f \equiv 0$. Periodic orbits of the vector field R correspond to periodic orbits of the map ϕ.

In particular, if $\Sigma = T^2$ and ϕ is an irrational rotation, then R has no periodic orbits. But this is a rare exception:

Theorem (with Taubes, 2008)

If Y is a closed oriented connected three-manifold which is not a T^2-bundle over S^1, then for every stable Hamiltonian structure on Y, the associated "Reeb vector field" R has a periodic orbit.

More general vector fields???

Part 3

We now review embedded contact homology (ECH), which is the main tool which we use to prove our theorems. This inputs a contact three-manifold (Y,λ) and outputs a homology which keeps track of the Reeb orbits and certain holomorphic curves between them.

Embedded contact homology

Embedded contact homology depends on:

- A closed 3-manifold Y.
- A nondegenerate contact form λ on Y.
- A homology class $\Gamma \in H_1(Y)$.

$ECH(Y, \lambda, \Gamma)$ is the homology of a chain complex over $\mathbb{Z}/2$. (One can also use $\mathbb{Z}$ coefficients).

A chain complex generator is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$ where:

- The α_i's are distinct embedded Reeb orbits.
- The m_i's are positive integers.
- $\sum_i m_i[\alpha_i] = \Gamma \in H_1(Y)$.
- $m_i = 1$ when α_i is hyperbolic (i.e. the linearized return map has real eigenvalues).

The differential

Choose a generic almost complex structure J on $\mathbb{R} \times Y$ such that:

- $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate.
- $J(\xi) = \xi$ and $d\lambda(v, Jv) \geq 0$ for $v \in \xi$.
- J is $\mathbb{R}$-invariant.

If α and β are two chain complex generators, the differential coefficient $\langle \partial\alpha, \beta \rangle \in \mathbb{Z}/2$ counts J-holomorphic curves C in $\mathbb{R} \times Y$, modulo the $\mathbb{R}$ action, which are asymptotic (as currents) to α as $s \to +\infty$ and to β as $s \to -\infty$, and which have "ECH index" $I(C) = 1$.

The precise definition of the ECH index I is a bit complicated. It turns out that the condition $I(C) = 1$ forces C to be embedded (except possibly for multiple covers of $\mathbb{R}$-invariant cylinders), hence the name "embedded contact homology".

Theorem (with Taubes, 2007)

$\partial^2 = 0$.

The U map

- There is also a degree 2 map

$$U : ECH_*(Y,\lambda,\Gamma) \to ECH_{*-2}(Y,\lambda,\Gamma).$$

  This is induced by a chain map which counts J-holomorphic curves with ECH index 2 that pass through a base point in $\mathbb{R}\times Y$. The U map is well-defined when Y is connected.
- The U map is very important in applications, because it is known to be nontrivial, and thus leads to the existence of many holomorphic curves with geometric consequences.

Isomorphism with Seiberg-Witten Floer cohomology

Theorem (Taubes, 2008)

If Y is connected, then there is a canonical isomorphism of relatively graded modules

$$ECH_*(Y, \lambda, \Gamma) \simeq \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + \Gamma).$$

Here $\widehat{HM}^*$ denotes Seiberg-Witten Floer cohomology as defined by Kronheimer-Mrowka, and $\mathfrak{s}_\xi + \Gamma$ is a spin-c structure determined by ξ and Γ. This isomorphism identifies the U map on ECH with an analogous map on $\widehat{HM}^*$.

In particular, ECH is a topological invariant of Y, ξ and Γ.
In fact, both sides have absolute gradings by homotopy classes of oriented plane fields, and the isomorphism respects these absolute gradings (Cristofaro-Gardiner, 2012), so one can write the isomorphism as $ECH_p(Y, \lambda) \simeq \widehat{HM}^p(Y)$ where p is a homotopy class of oriented plane fields.

Some consequences of the isomorphism

Using properties of $\widehat{HM}$ due to Kronheimer-Mrowka, we deduce the following:

- $ECH_*(Y,\lambda,\Gamma)$ is finitely generated in each grading $*$.

If $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ is torsion (such Γ always exist), then $ECH_*(Y,\lambda,\Gamma)$ has a noncanonical $\mathbb{Z}$-grading, and:

- $ECH_*(Y,\lambda,\Gamma)$ vanishes when the grading $*$ is sufficiently small
- $ECH_*(Y,\lambda,\Gamma)$ is nonzero in infinitely many gradings
- If the grading $*$ is sufficiently large then

$$U : ECH_*(Y,\lambda,\Gamma) \to ECH_{*-2}(Y,\lambda,\Gamma).$$

Deducing the Weinstein conjecture

Corollary

If λ is nondegenerate, and if $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ is torsion, then there exists a nonempty ECH generator $\alpha = \{(\alpha_i, m_i)\}$ with $\sum_i m_i[\alpha_i] = \Gamma$.

Proof. We know that $ECH_*(Y, \xi, \Gamma)$ is infinitely generated. If there is no nonempty ECH generator in the class Γ, then

$$ECH_*(Y,\xi,\Gamma) = \begin{cases} \mathbb{Z}/2, & \Gamma = 0, \\ 0, & \Gamma \neq 0. \end{cases}$$

which is a contradiction. (Here the $\mathbb{Z}/2$ is generated by the empty set of Reeb orbits.) □

Filtered ECH

If $\alpha = \{(\alpha_i, m_i)\}$ is an ECH generator, define its symplectic action by

$$\mathcal{A}(\alpha) = \sum_i m_i \int_{\alpha_i} \lambda.$$

It follows from the conditions on J that the differential decreases symplectic action: if $\langle \partial\alpha, \beta \rangle \neq 0$ then $\mathcal{A}(\alpha) > \mathcal{A}(\beta)$. Thus we can make:

Definition

Given $L \in \mathbb{R}$, we can define $ECH^L(Y, \lambda, \Gamma)$, called filtered ECH, to be the homology of the subcomplex generated by α with $\mathcal{A}(\alpha) < L$.

Unlike the usual ECH, filtered ECH depends heavily on the contact form λ.

The ECH spectrum

Definition

If $0 \neq \sigma \in ECH(Y, \xi, \Gamma)$, define $c_\sigma(Y, \lambda)$ to be the infimum of L such that σ is in the image of the map $ECH^L(Y, \lambda, \Gamma) \to ECH(Y, \xi, \Gamma)$.

The numbers $c_\sigma(Y, \lambda)$ are the "ECH spectrum" of (Y, λ).

Remark

By definition, there exists an ECH generator with symplectic action equal to $c_\sigma(Y, \lambda)$. So if $\sigma \neq [\emptyset]$, then there exists a Reeb orbit of action less than or equal to $c_\sigma(Y, \lambda)$.

One can also define $c_\sigma(Y, \lambda)$ when λ is degenerate by taking a limit of nondegenerate contact forms.

Example: the boundary of an ellipsoid

- Suppose $Y = \partial E(a, b)$ with its canonical contact form λ.
- If a/b is irrational then λ is nondegenerate: the linearized return map around γ_1, respectively γ_2, is rotation by $2\pi a/b$, respectively $2\pi b/a$.
- The chain complex generators have the form $\gamma_1^a \gamma_2^b$ with $a, b \in \mathbb{N}$. The symplectic action of such a generator is $am + bn$.
- The differential vanishes because every generator has even grading.
- So in this case the numbers in the ECH spectrum are the nonnegative linear combinations of a and b.
- If we write the generators in order of increasing symplectic action as $\zeta_0, \zeta_1, \zeta_2, \ldots$ then $U\zeta_k = \zeta_{k-1}$ for $k > 0$ and $U\zeta_0 = 0$.

Part 4

We now outline how embedded contact homology can be used to prove the existence of two Reeb orbits (and a bit more) for any contact form on a closed three-manifold.

Asymptotics of the ECH spectrum

The key result is the following:

Theorem (with D. Cristofaro-Gardiner and V. Ramos, 2012)

Let (Y, λ) be a closed connected contact three-manifold, possibly degenerate. Let $\{\sigma_k\}_{k \geq 1}$ be a sequence in $ECH(Y, \xi, \Gamma)$ with $U\sigma_{k+1} = \sigma_k$. (Such a sequence always exists by the isomorphism with $\widehat{HM}$.) Then

$$\lim_{k \to \infty} \frac{c_{\sigma_k}(Y, \lambda)^2}{k} = 2\,\mathrm{vol}(Y, \lambda).$$

Recall that $c_{\sigma_k}(Y, \lambda)$ is the length of a finite set of (possibly multiply covered) Reeb orbits. So this theorem relates the volume of (Y, λ) to the lengths of certain distinguished sets of Reeb orbits.

Example: the boundary of an ellipsoid

Let us check this theorem when $Y = \partial E(a, b)$, and $\sigma_k = \zeta_k$.

- $c_{\zeta_k}(\partial E(a, b)) = L$ where the triangle bounded by the x axis, the y axis, and the line $ax + by = L$ encloses $k + 1$ lattice points.
- This triangle has area $L^2/2ab$, so
$$k = \frac{L^2}{2ab} + O(L).$$
- Thus
$$\lim_{k\to\infty} \frac{c_{\zeta_k}(\partial E(a, b))^2}{k} = 2ab.$$
- It is an exercise to check that
$$\mathrm{vol}(\partial E(a, b)) = ab.$$

A key lemma

Lemma

Let (Y, λ) *be a closed connected contact three-manifold. Suppose there are only finitely many embedded Reeb orbits* $\gamma_1, \ldots, \gamma_k$*. Then:*

- *For any* $\sigma \in ECH_*(Y, \xi, \Gamma)$*, the number* $c_\sigma(Y, \lambda)$ *is a nonnegative integer linear combination of* $\mathcal{A}(\gamma_1), \ldots, \mathcal{A}(\gamma_k)$*.*
- *If* $U\sigma \neq 0$*, then* $c_{U\sigma}(Y, \lambda) < c_\sigma(Y, \lambda)$*.*

If λ is nondegenerate this follows directly from the definitions. The proof when λ is degenerate requires a limiting and Gromov compactness argument.

Remark

If there are infinitely many embedded Reeb orbits, then it is possible that $c_{U\sigma}(Y, \lambda) = c_\sigma(Y, \lambda)$. For example this happens sometimes when $Y = \partial E(a, b)$ and a/b is rational.

Proof of the existence of two embedded Reeb orbits

- Let (Y, λ) be a closed connected contact three-manifold. Suppose there is only one embedded Reeb orbit γ, with action T.
- Choose $\Gamma \in H_1(Y)$ such that $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ is torsion, and let $\{\sigma_k\}_{k\geq 1}$ be a sequence of nonzero classes in $ECH(Y, \lambda, \Gamma)$ with $U\sigma_{k+1} = \sigma_k$.
- By the Key Lemma, $c_{\sigma_k}(Y, \lambda) = m_k T$ where m_k is a nonnegative integer and $m_k < m_{k+1}$.
- Therefore

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)}{k} \geq T.$$

- This contradicts the theorem on the asymptotics of the ECH spectrum which asserts that

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{k} = 2\,\mathrm{vol}(Y,\lambda) < \infty.$$

Relation to the short Reeb orbit conjecture

Theorem (same collaborators)

Let (Y,λ) be a closed connected contact three-manifold. Then at least one of the following is true:

- *There exists a Reeb orbit γ with $\mathcal{A}(\gamma)^2 \le \mathrm{vol}(Y,\lambda)$.*
- *There exist at least three embedded Reeb orbits.*

To prove this, we will show that if there are only two embedded Reeb orbits γ_1 and γ_2, with actions T_1 and T_2, then

$$T_1 T_2 \le \mathrm{vol}(Y,\lambda).$$

To do so, let $\{\sigma_k\}_{k\ge 1}$ be as before. Then $c_{\sigma_k}(Y,\lambda) = m_k T_1 + n_k T_2$ with $m_k, n_k \in \mathbb{N}$ and $m_k T_1 + n_k T_2 < m_{k+1} T_1 + n_{k+1} T_2$. As in the ellipsoid example, it follows that

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{k} \ge 2 T_1 T_2.$$